Distributed Policy Optimization under Partial Observability: Tractability, Linear Speedup, and Communication Efficiency

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October 2024

Abstract

We consider the problem of policy optimization within the context of Partially Observable Markov Decision Processes (POMDPs) in a distributed setting where M clients collaborate under the coordination of a central server. We develop a new theoretical framework to characterize the performance of distributed policy optimization for POMDPs. Specifically, we propose a novel actor-critic framework, where the agents collaboratively perform policy improvement and evaluation to maximize the benefit of parallelization. Under linear function approximation, we rigorously establish global convergence rates for a wide range of softmax policies, achieving linear speedup in sample complexity along with sublinear communication complexity. Our analysis builds on a novel, sharp bound for local drift terms and a parameter stacking technique to analyze temporal correlations that are of independent interest in studying POMDPs.

Keywords: Partially-observable Markov Decision Process, linear speedup, communication efficiency

Contents

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1 Introduction

Reinforcement learning (RL) is a sequential decision-making process where an agent aims to maximize her cumulative reward through repeated interactions with an unknown environment [47]. RL is described by the Markov Decision Process (MDP) [40], in which the agent chooses an action at each moment based on the current state and receives a reward. The environment then transitions to a new state in response to this action. An essential assumption in MDPs is that the agent has complete knowledge of the environment's state at every moment. However, in many practical applications, such as predicting stock prices [23] and training humanoid robots [27], the agent has access only to unreliable or incomplete information about the environment's state. We model these sequential decision-making problems as Partially Observable Markov Decision Processes (POMDPs) [36]. In a POMDP, unlike an MDP, the agent takes action based solely on noisy *observations* of the states.

While POMDPs provide significant modeling flexibility, theoretical investigations into their tractability present a discouraging outlook. [38] and [21] have established that obtaining the optimal policy for POMDPs using exact methods, such as dynamic programming, is computationally and statistically intractable. Furthermore, existing RL theories for POMDPs often emphasize learning the transition dynamics while assuming access to a perfect planning algorithm, whose computational complexity is prohibitive in large observation space [17, 21, 30]. Despite these discouraging theoretical results, POMDP algorithms based on deep reinforcement learning have achieved tremendous empirical success [34, 59]. The adoption of neural networks enables the extraction of structured representations from historical observations, which is advantageous for policy optimization, albeit at the cost of an additional approximation error [14]. More importantly, empirical methods have also demonstrated the benefits of parallelization in the policy optimization of POMDPs [10, 42, 48] (see Fig 1 for an illustration).

In this work, we take the first step to bridge this gap between theory and practice by developing new theoretical results that explain the empirical success of POMDPs, particularly in the parallel setting.

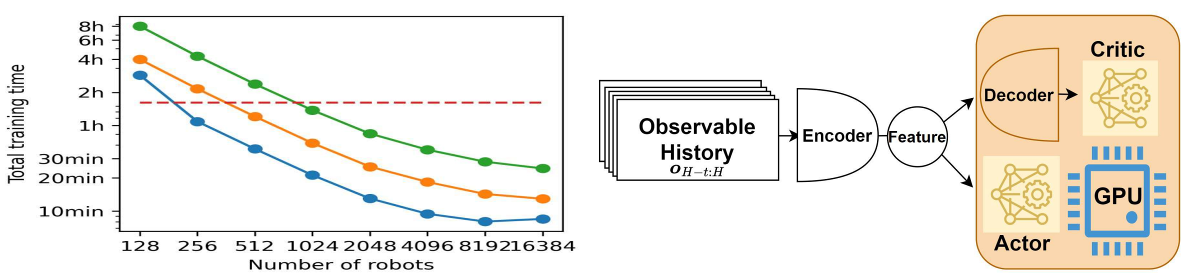

Figure 1: The left figure, adapted from Figure 4(b) in [42], demonstrates that partially observable RL is consistently accelerated when more agents are trained in parallel, even though it involves greater complexity than MDPs. This phenomenon is what our theory seeks to explain. The right figure illustrates practical solutions for POMDPs, where actor and critic networks carry out massively parallel policy optimization after processing feature representations extracted from historical inputs. The colored region highlights the procedure we aim to analyze.

1.1 Contributions

We study the problem of distributed reinforcement learning in partially observable environments, where multiple agents collaboratively learn the optimal policy of a shared, underlying POMDP. We develop novel theoretical results that explain the success of practical POMDP algorithms in distributed setups. We demonstrate how these methods trade the computation complexity off with extra error in function approximation and optimization, which is the key to their scalability. We also show why policy optimization for POMDP benefits from data parallelism, similar to MDP, and where the fundamental differences lie in their convergence rates. Our contributions include:

- *Algorithm:* We propose algorithms to model practical policy evaluation and improvement for POMDPs, preserving key ingredients of distributed optimization, such as client-server communication, multiple local steps, stochastic gradients, and weighted averaging.
- *Complexity:* We establish global convergence rates for a wide range of softmax policies. Under linear function approximation, we show that our algorithms enjoy linear speedup in sample complexity and a communication complexity that is sublinear in the computation cost per agent. These terms, as well as the computation complexity, are independent of the sizes of the state and observation space.
- *Technique:* We offer new methods to analyze distributed POMDP training from a theoretical standpoint and a technique to bound the local drift effect with Riemannian integration.

1.2 Related Work

Partially-observable RL Existing POMDP theories primarily focus on learning transition dynamics for subclasses of POMDPs with polynomial sample complexity [8, 31]. Such studies usually assume access to black-box planning algorithms [21, 28, 30], which outputs the value function and the corresponding optimal policy, given estimated transition kernels. Their analysis crucially relies on a perfect planner. While few works advance these results with detailed designs for the planning oracles [16, 18], their methods involve an exhaustive search over all possible sample paths of observations, consuming computation and memory resources that scale exponentially with the size of the observation space. Recent studies [9, 58] improve the tractability of theoretical POMDP algorithms with parameterized policies. However, they did not analyze the sample complexity in policy optimization and failed to explain how parallel training accelerates this process. Our work stands out distinctly from these results. We provide a finite-time sample complexity analysis on the error in the planning procedure, which is irrelevant to the size of observation space, and quantitatively characterize how our algorithm benefits from massive parallel training, aligning with the empirical phenomenon observed in [42].

Algorithm	Continuous Observation	Exhaustive Search	Planning Error Analysis	Distributed Training
OMLE [30]	✗	✓	N/A	✗
PORL^2 [19]	✗	✓	N/A	✗
μLV-Rep [58]	✓	✗	N/A	✗
DPAC (Ours, Alg. 1)	✓	✗	✓	✓

Table 1: Comparison of representative theoretical POMDP algorithms with Algorithm 1. "Exhaustive search" refers to an ideal planning process that finds the optimal policy by taking the maximum argument of the value function or exploring all possible histories using dynamic programming.

Distributed Policy Optimization. Recent distributed (federated) policy gradient methods mainly focus on Markov policies [52] with tabular softmax parameterization [56]. Their data parallel routine synchronizes every iteration [26], resulting in high communication costs. We extend previous results to history-dependent policies with general softmax parameterization, while allowing multiple local steps between updates. Table 2 presents a comparison between these studies.

Algorithm	Sample Complexity	Global Convergence	Linear Speedup	Local Steps	Partial Observation
FedNPG-ADMM [26]	$O\left(\frac{1}{(1-\gamma)^6 M\epsilon^2}\right)$	✗	✗	✗	✗
FedNPG [56]	$O\left(\frac{(1-\gamma)^{-11.5}}{\sqrt{M}\epsilon^{\frac{7}{2}}}\right)$	✓	✗	✗	✗
FEDHAPG-M [52]	$O\left(\frac{1}{M\epsilon^{3/2}}\right)$	✗	✓	✓	✗
DPPG (Ours, Alg. 3)	$O\left(\frac{H^4}{M\epsilon^4}\right)$	✓	✓	✓	✓

Table 2: Comparison of our policy improvement algorithm 3 with distributed policy gradient algorithms for MDPs. d_θ denotes the dimension of the policy's parameters, M is the number of parallel machines and T is the number of gradient computation per machine.

Many existing works on distributed (federated) TD learning focus on a decentralized manner [12] or did not contain a linear speedup analysis in their sample complexity [32]. Our policy evaluation method extends recent TD learning studies [49, 53] from MDPs to POMDPs, ensures sublinear communication complexity, and relaxes the condition for achieving linear speedup in sample complexities, which is crucial to massively parallel training. See Table 3 for a comparison.

Algorithm	Sample Complexity	Speedup Condition	Communication Complexity	Partial Observation
FedTD(0) [53]	$O\left(\frac{d_Q^2}{\nu_\phi{}^2 M\epsilon}\right)$	$M \ll T$	N/A	✗
TD(0) with local state [49]	$\widetilde{O}\left(\frac{1}{\nu_\phi{}^2 M\epsilon}\right)$	$M \ll T$	One-shot	✗
DPTD (Ours, Alg. 2)	$\widetilde{O}\left(\frac{\frac{H\sigma_g^2}{\nu_\phi}+Hd_Q}{M\epsilon}\right)$	Allow for Large M	Sublinear in T	✓

Table 3: Comparison between our policy evaluation algorithm 2 and distributed TD learning algorithm for MDPs. ν_ϕ is the condition number of linear function approximation.

2 Problem Formulation

Notations. We use $\lesssim$ or $\widetilde{O}$ to omit constants or logarithmic factors in the expressions, respectively. For $p \in [1, \infty]$ and $v, w \in \mathbb{R}^d$, $\|v\|_p$ denotes the p-norm of the vector v and $\langle v, w\rangle_{\mathbb{R}^d}$ denotes the inner product between the vectors v and w. We use $\Delta(\mathcal{X})$ to denote the probability simplex over a finite set $\mathcal{X}$. For matrices A and B, the notation $A \succeq B$ implies $A - B$ is positive semi-definite. We denote the Moore-Penrose pseudo-inverse of a matrix A using $A^\dagger$. Lastly, for probability distributions P and Q, $D_{KL}(P||Q) = \mathbb{E}_{X\sim P}\left[\log \frac{p(X)}{q(X)}\right]$ denotes the Kullback-Leibler (KL) divergence between P and Q.

The POMDP Model. We consider an episodic Partially Observable Markov Decision Process with a state space $\mathcal{S}$, action space $\mathcal{A}$, observation space $\mathcal{O}$ and a discount factor $\gamma \in (0,1)$ over a finite horizon H. We assume that the action space is finite, while state and observation spaces can be potentially continuous sets. A POMDP is associated with a deterministic reward function $r_h : \mathcal{S} \times \mathcal{A} \to [0,1]$, transition kernel $\mathbb{T}_{h,a}(\cdot|s) : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$ and emission matrix $\mathbb{O}_h(\cdot|s) : \mathcal{O} \to \mathcal{S}$ for all $h \in \{0, 1, \ldots, H-1\}$. At any time h, let s_h denote the state of the environment. When the agent takes an action a_h, it receives a reward $r_h(s_h, a_h)$ and results in the environment to transition into state $s_{h+1} \sim \mathbb{T}_{h,a_h}(\cdot|s_h)$. The agent then observes $o_{h+1} \sim \mathbb{O}_{h+1}(\cdot|s_{h+1})$, which is used to take the next action a_{h+1}. The initial state of the POMDP S_0 is drawn from a distribution ρ over the state space $\mathcal{S}$.

Let $Z_h := (O_0, A_0, \ldots, O_{h-1}, A_{h-1}, O_h) \in \mathcal{Z}_h$ denote the set of observations and actions till time h, which we refer to as the observable "history". We also name $\mathcal{Z}_h := (\mathcal{O} \times \mathcal{A})^h \times \mathcal{O}$ as the history space at time h. A policy π is collection of mappings $\pi_h : \mathcal{Z}_h \to \Delta(\mathcal{A})$ for all $h \in \{0, 1, \ldots, H-1\}$, where π_h denotes a rule for selecting actions at time h based on the history Z_h such that $\pi_h(a_h|Z_h)$ is the probability of choosing the action a_h given the history Z_h. For the simplicity of notation, we use the shorthand $\bar{z}_h$ to denote (z_h, a_h). Similar to MDPs, a policy π is associated with a value function and a Q-function, which we define as follows [9]:

$$\mathsf{V}_t^\pi(z_t) := \mathbb{E}^\pi\left[\sum_{h=t}^{H} \gamma^{h-t} r_h(S_h, A_h) \big| Z_t = z_t\right]; \quad \mathsf{Q}_t^\pi(z_t, a_t) := \mathbb{E}^\pi\left[\sum_{h=t}^{H} \gamma^{h-t} r_h(S_h, A_h) \big| \bar{Z}_t = (z_t, a_t)\right]. \tag{1}$$

We also define an advantage function associated with a policy π, which is given by

$$\mathsf{A}_t^\pi(z_t, a_t) := \mathsf{Q}_t^\pi(z_t, a_t) - \mathsf{V}_t^\pi(z_t)$$

The learning objective is to find the policy π from a given class of policies Π that maximizes the expected reward, i.e.,

$$\sup_{\pi \in \Pi} \mathbb{E}^\pi\left[\sum_{h=0}^{H-1} \gamma^h r_h(S_h, A_h)\right]. \tag{2}$$

Policy Parametrization. For the convenience of policy optimization over a large observation space, we associate the policies with a parameter $\theta \in \mathbb{R}^{d_\theta}$. When the action space is finite, we optimize the policies within the class of history-dependent softmax policies [1], which take the form of

$$\pi_h^\theta(a_h|z_h) := \frac{\exp f_\theta(z_h, a_h; h)}{\sum_{a_h \in \mathcal{A}} \exp f_\theta(z_h, a_h; h)} \tag{3}$$

Here, $f_\theta(\cdot, \cdot; h) : \mathcal{Z}_h \times \mathcal{A} \to \mathbb{R}$ is a real-valued function for any time index h. Eq. (3) encompasses a wide range of policies, including softmax policies with tabular representation, where $f_\theta(z_h, a_h; h) = \theta(z_h, a_h, h)$; "log-linear policies," where $f_\theta(z_h, a_h; h) = \theta^\top \xi_h(z_h, a_h)$ and $\xi_h(z_h, a_h)$ is a known feature with dimension d_θ; and "neural softmax policies," where $f_\theta(\cdot, \cdot; \cdot)$ is a non-linear sequential neural network.

Linear Function Approximation. Following the convention of theoretical RL [6, 35, 50], we adopt linear function approximation for the Q-functions in a continuous observation space. We assume that the Q-function of any policy π can be represented by a set of vectors $\{\psi_h^\pi\}_{h=0}^{H-1}$ as follows: $\mathsf{Q}_h^\pi(z_h, a_h) = \langle \phi_h(z_h, a_h), \psi_h^\pi \rangle$ for all $(z_h, a_h) \in \mathcal{Z}_h \times \mathcal{A}$ and $h \in \{0, 1, \ldots, H-1\}$. Here, vectors $\{\psi_h^\pi\}_{h=1}^{H}$ encode the information of the policy, while $\{\phi_h\}_{h=0}^{H-1}$ denotes a collection of known functions where $\phi_h : \mathcal{Z}_h \times \mathcal{A} \to \mathbb{R}^{d_Q}$. These functions $\{\phi_h\}_{h=1}^{H}$ form a compact representation of the POMDP dynamics at time h. A rich subset of POMDPs fall satisfies this assumption, including tabular POMDPs, where value functions are the inner product between the α-vectors and belief states [39] with $d = |\mathcal{S}|$; L-decodable POMDPs [13] with rank d_{rank} transitions, where $Q_h(z_h, a_h) = p_h^\top \omega_h^\pi$ and $d = d_{\text{rank}}$ (see Eq. (11) in [58], Prop 2.3 in [22] for details).

Remark 1. In this work, we primarily focus on policy optimization, assuming that the features are already known. The problem of learning good feature representation has been well-studied for a various class of POMDPs [18, 51, 58].

Distributed Optimization We consider reinforcement learning in a distributed (federated) regime, where M agents collaboratively optimize a shared POMDP under the coordination of a central server. Each agent independently optimizes their local parameters for K iterations, after which the central server aggregates the parameters to update a central policy. The agents then synchronize their policies and begin the next phase of local iterations, and we repeat this process for R rounds. We use $T = RK$ to denote the total computation budget per agent.

3 Algorithm Design

We present an actor-critic algorithm in which multiple agents solve a shared POMDP in a collaborative manner. We refer to the algorithm as "Distributed Partially Observable Natural Actor Critic" (Alg. 1), or "DPAC" for short.

Algorithm 1: DPAC

Initialize actor parameter θ_0, critic parameter ψ_0.
for round r in $0, 1, \ldots, R-1$ **do**
  Send θ_r, ψ_r to client agents
  //Critic: policy evaluation
  Evaluate Q-function $\widehat{\mathsf{Q}}^{\pi^{\theta_r}}$ by running Alg 2 for T_e steps.
  Send $\widehat{\mathsf{Q}}^{\pi^{\theta_r}}$ to client agents
  //Actor: policy improvement
  Approximate policy gradient ω_r by running line 6 to 17 in Alg 3 with K local steps.
  Update policy in central server $\theta_{r+1} \leftarrow \theta_r + \eta_r \cdot \omega_r$
end for
return Uniform mixture of $\theta_0, \theta_1, \ldots, \theta_{R-1}$

DPAC operates by alternately performing policy evaluation and improvement to optimize a shared policy. At each iteration r, DPAC transmits the central policy, parameterized by θ_r, to parallel machines. During this phase, each agent's critic collaboratively estimates the Q-function corresponding to the central policy π^{θ_r} using an algorithm referred to as "Distributed Partially-observable Temporal Difference Learning" (Alg. 2), abbreviated as "DPTD." After we aggregate the Q-function in the central server, DPAC executes another routine named "Distributed Partially-observable Natural Policy Gradient" (Alg. 3), or "DPPG" for short. In this step, multiple agents compute the policy gradient in parallel, and the results are subsequently combined at the central server, leading to an update of the central policy. This iterative process continues for a total of R rounds, after which DPAC returns a uniform mixture of the policies parameterized by $\{\theta_0, \theta_1, \ldots, \theta_{R-1}\}$.

In the following sections, we provide a detailed description of our policy evaluation and improvement routines.

3.1 Policy Evaluation

The policy evaluation algorithm estimates the value function of a given policy. While this process is well-understood for MDPs [1], designing a distributed pipeline for POMDPs presents three additional challenges.

First, as [58] notes, POMDPs' history-dependence leads to computational bottlenecks. Exact evaluation of the value functions involves an exhaustive search through the history space $\mathcal{Z}_H$ to calculate a marginalization constant, bringing computational and spatial costs that scale with $(\mathcal{O}\mathcal{A})^H$. Second, unlike infinite-horizon stationary MDPs, the Bellman operator for finite-horizon POMDPs is naturally time-variant, potentially resulting in H consecutive regression problems [58]. However, optimization literature indicates [25] that errors in a series of regressions could compound, hindering convergence and complicating finite-time analysis. Furthermore, optimizing non-Markovian policies in finite horizon POMDPs requires preserving the temporal correlation in transition tuples instead of sampling from the same stationary distribution in MDPs. Thirdly, addressing the evaluation problem in parallel settings presents additional challenges, such as correcting the cumulative bias in local updates.

To address the first problem, we introduce linear function approximation for the value functions, simplifying the structure of the estimated Q-function. This transformation converts the dynamic programming problem into a least-squares regression that minimizes the Bellman error [47] at each time step:

$$\underset{\{\psi_h\}}{\text{minimize}}\ \frac{1}{2}\mathbb{E}^{\pi}\left[\left(\mathsf{Q}_h^{\pi}-\langle\phi_h(\bar{Z}_h),\psi_h\rangle\right)^2\right]\ \text{ for all } h\in\{0,1,\dots,H-1\} \tag{4}$$

The features ϕ_h provide a compact representation of POMDP transition dynamics, freeing us from computing a complex normalizing constant in the Bellman backup. To avoid exhaustive history searches, we adapt the Temporal-difference learning algorithm from MDPs to POMDPs [9], which updates the Q-function by bootstrapping from its current estimate, thus allowing for direct optimization of the parameter ψ_h without acquiring full knowledge of Q_{h+1}.

To tackle the second challenge, we stack the parameter estimates ψ_h across all time steps and jointly optimize the H consecutive Bellman errors. This approach prevents compounding error and enables parallel updates of $\{\psi_h\}_{h=0}A^{H-1}$, thereby speeding up optimization. Specifically, we define $\psi=(\psi_0^\top,\psi_1^\top,\dots,\psi_{H-1}^\top)^\top\in\mathbb{R}^{Hd_Q}$ as the stacked vector of parameters, and we solve the following optimization problem:

$$\underset{\psi}{\text{minimize}}\ \mathcal{E}^{\pi}(\psi),\ \text{ where } \mathcal{E}^{\pi}(\psi):=\frac{1}{2}\cdot\mathbb{E}\left[\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{Q}_h^{\pi}-\langle\phi_h(\bar{Z}_h),\psi_h\rangle\right)^2\right]$$

To preserve the temporal correlation in the data, we sample entire roll-out trajectories $\bar{Z}_H$ from the POMDP model instead of individual transition tuples, as done in the MDP context. We then extract the first h tuples to create $\bar{Z}_{h+1}$, which is used to compute the gradient for ψ_h:

$$g_h(\psi;\bar{Z}_{h+1}):=-\gamma^h\phi_h(\bar{Z}_h)\cdot\left(r_h+\gamma\phi_{h+1}(\bar{Z}_{h+1})^\top\psi_{h+1}-\phi_h(\bar{Z}_h)^\top\psi_h\right) \tag{5}$$

The term $g_h(\psi;\bar{Z}_{h+1})$ is the semi-gradient [47]. We then stack g_h throughout the time horizon:

$$g(\psi;\bar{Z}_H):=\left[g_0(\psi;\bar{Z}_1)^\top,\dots,g_{H-1}(\psi;\bar{Z}_H)^\top\right]^\top$$

which are employed to update the Q-function estimates at each time instant, all at once,

$$\psi^{r,k+1,m}\leftarrow\psi^{r,k,m}-\alpha_r\cdot g(\psi^{r,k,m};\bar{Z}_H), \tag{6}$$

Here, α_r is the local learning rate in round r.

We solve the third challenge by carefully designing the data parallel routine. Each agent performs K local updates and then sends its estimate $\psi^{r,K,m}$ to the server. The server averages these estimates to produce $\widetilde{\psi}^{r+1}$, which helps reduce variance across local machines. Agents then begin the next round of updates from $\psi^{r+1,0,m}=\widetilde{\psi}^{r+1}$ for all m. We select a sufficiently small local learning rate α_r to minimize the deviation of local parameters from their recent synchronization $\widetilde{\psi}^{r+1}$. However, a smaller α_r can also slow down convergence. Drawing inspiration from [29, 53], we implement a weighted averaging strategy at the central server:

$$\widetilde{\psi}^{r+1}:=(1-\eta_r)\widetilde{\psi}^{r+1}+\eta_r\frac{1}{M}\sum_{m=1}^{M}\psi^{r,K,m}=\widetilde{\psi}^{r}-\eta_r\alpha_r\cdot\frac{\sum_m}{M}\sum_{k=0}^{K-1}g_h(\psi^{r,k,m};\bar{z}_{h+1}^{r,k,m})$$

Since the gradients in the server possess less noise, it is safe to choose a larger global step size η_r for faster convergence. While the bootstrapping technique increases the flexibility of the TD learning algorithm, it also introduces bias into the gradient estimates [47]. Fortunately, this bias diminishes as the estimation error decreases. This insight inspired us to discard inaccurate Q-function estimates from early iterations and conduct a weighted averaging during the last $R-R_0$ rounds, using quadratically increasing weights ω_r. This process allows us to produce the final Q-function estimate as:

$$\widehat{\psi}=\sum_{r=R_0}^{R}\omega_r\widetilde{\psi}^{r}$$

where R_0 is a carefully chosen parameter, specified in Theorem 1.

3.2 Policy Improvement

Given the policy evaluation routine detailed in Section 3.1, we update the policy parameter by a variant of the Natural Policy Gradient (NPG) algorithm [24]. NPG is a model-free RL algorithm with solid theoretical justification, and it underpins many practical RL algorithms, such as TRPO [43] and PPO [44]. Building upon previous works [4, 9], it is straightforward to adapt the NPG algorithm to POMDPs.

Algorithm 2: DPTD

Input policy π, feature $\{\phi_h\}_{h=0}^{H-1}$, local l.r. $\{\alpha_r\}_{r=0}^{R-1}$, global l.r. $\{\eta_r\}_{r=0}^{R-1}$, weights $\{w_r\}_{r=0}^{R}$.
Initialize $\psi^{0,0,m} \leftarrow 0$, $\widetilde{\psi}^r \leftarrow 0$.
for round $r = 0, 1, \ldots, R-1$ **do**
  //Distributed training
  for machine $m = 0, 1, \ldots, M-1$ in parallel **do**
    for local step $k = 0, 1, \ldots, K-1$ **do**
      Sample $\bar{Z}_H^{r,k,m}$ from π
      Compute $g(\psi^{r,k,m}; \bar{Z}_H^{r,k,m})$ via Eq. (5)
      $\psi^{r,k+1,m} \leftarrow \psi^{r,k,m} - \alpha_r g\left(\psi^{r,k,m}; \bar{Z}_H^{r,k,m}\right)$
    end for
  end for
  Clients send $\psi^{r,K,m}$ to server
  //Centralized processing
  Reduce variance $\bar{\psi}^{r+1} \leftarrow \frac{1}{M}\sum_{m=0}^{M-1} \psi^{r,K,m}$
  Poliak averaging $\widetilde{\psi}^{r+1} \leftarrow (1-\eta_r)\,\widetilde{\psi}^r + \eta_r \bar{\psi}^{r+1}$
  Synchronize $\psi^{r+1,0,m} \leftarrow \widetilde{\psi}^{r+1}$
  if $r \geq R_0$: **then**
    Weighted sum $\widehat{\psi} = \widehat{\psi} + w_{r+1} \cdot \widetilde{\psi}^{r+1}$
  end if
end for
return $\{\widehat{Q}_h(\cdot) = \langle \phi_h(\cdot)\ ,\ \widehat{\psi}_h \rangle_{\mathbb{R}^{d_Q}}\}_{h=0}^{H-1}$

Algorithm 3: DPPG

Input θ_0, evaluation procedure PolicyEval$(\cdot)$
for round r in $0, 1, \ldots, R-1$ **do**
  Evaluate policy in server $\mathsf{Q} \leftarrow \text{PolicyEval}\left(\pi^{\theta_r}\right)$
  Send θ_r, Q to client agents.
  //Estimate gradient ω by distributed training
  for machine m in $1, 2, \ldots, M$ in parallel **do**
    Initialize $\omega_r^{0,m} \leftarrow 0$
    for local step k in $0, 1, \ldots, K-1$ **do**
      Collect trajectory $\bar{Z}_H^{k,m}$ by playing π^{θ_r}
      $\omega_r^{k+1,m} \leftarrow \omega_r^{k,m} - \zeta_k \nabla_\omega \widehat{L}_{\mathsf{Q}}\left(\omega_r^{k,m}; \theta_r, \bar{Z}_H^{k,m}\right)$
      //Aggregate ω in server
      if $k \equiv 0 \pmod{I}$ **then**
        $\omega_r^{k+1} \leftarrow \frac{1}{M}\sum_{m=1}^{M} \omega_r^{k+1,m}$
        Synchronize $\omega_r^{k+1,m} \leftarrow \omega_r^{k+1}$
      end if
    end for
  end for
  //Update policy in server
  Update policy by $\theta_{r+1} \leftarrow \theta_r + \eta_r \cdot \omega_r$
end for
return Uniform mixture of $\theta_0, \theta_1, \ldots, \theta_{R-1}$

NPG for POMDPs NPG updates policy parameter θ with

$$\theta_{t+1} \leftarrow \theta_t + \eta \cdot \mathcal{F}_\theta^\dagger \nabla_\theta \mathsf{V}^{\pi^{\theta_t}} \tag{7}$$

Here, $\mathcal{F}_\theta$ is the Fisher-information of policy π^θ, which is defined as

$$\mathcal{F}_\theta := \mathbb{E}^{\pi^\theta}\left[\sum_{h=0}^{H-1} \gamma^h \nabla_\theta \ln \pi_h^\theta(A_h|Z_H) \nabla_\theta^\top \ln \pi_h^\theta(A_h|Z_H)\right] \tag{8}$$

for POMDPs [9]. $\mathcal{F}_\theta$ corrects the direction of the gradient flow, resulting in faster convergence. For efficiency, we do not compute the pseudo-inverse in Eq. (7). Instead, we approximate the corrected gradient $\mathcal{F}_\theta^\dagger \nabla_\theta \mathsf{V}^{\pi^{\theta_t}}$ as a whole, according to a process called "compatible function approximation" [24]:

$$\omega^\star \in \underset{\omega \in \mathbb{R}^d}{\operatorname{argmin}}\ \mathbb{E}^{\pi^\theta}\left[\sum_{h=0}^{H-1} \gamma^h \left(\omega^\top \nabla_\theta \ln_h^{\pi^\theta}(A_h|Z_H) - \mathsf{A}_h^{\pi^\theta}(Z_h, A_h)\right)^2\right] \tag{9}$$

Eq. (9) results in $\theta_{t+1} \leftarrow \theta_t + \eta \cdot \omega_t$ and we refer to ω_t as the "NPG direction", or the NPG "policy gradient". This result is straightforward to extend to finite-horizon discounted POMDPs [9], in which we minimize the function

$$\mathcal{L}_{\mathsf{A}}(\omega; \theta_r, \pi^{\theta_r}) := \mathbb{E}^{\pi}\left[\sum_{h=0}^{H-1} \gamma^h \left(\omega^\top \nabla_\theta \ln \pi_h^\theta(A_h|Z_H) - \mathsf{A}_h^{\pi^{\theta_r}}(Z_H, A_h)\right)^2\right] \tag{10}$$

Q-NPG Eq. (9) adapts to a general policy class on continuous action spaces. However, it relies on a precise estimate of the advantage function, which results in high sample complexity [2]. When the action space is finite , we can alternatively optimize the NPG direction ω according to the "Q-NPG" update rule [1]. This method does not require an advantage function estimate, and we can instead estimate the Q-function with temporal difference learning introduced in Section 3.1. The Q-NPG rule obtains ω by minimizing the following function:

$$\mathcal{L}_{\mathsf{Q}}(\omega;\theta_r,\pi^{\theta_r}) := \mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}} \sum_{h=0}^{H-1} \gamma^h \left(\omega^\top \nabla_\theta f_{\theta_r}(Z_h, A_h; h) - \mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h)\right)^2 \tag{11}$$

We use parallelized linear regression [20] to approximate the policy gradient ω_r, using samples of the function $\widehat{\mathcal{L}}_{\mathsf{Q}}(\omega;\theta;\bar{Z}_H) := \sum_{h=0}^{H-1} \gamma^h \left(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega^\top \nabla_\theta f_{\theta_r}(Z_h, A_h; h)\right)^2$. This approach provides a distributed, model-free policy improvement algorithm for POMDPs (Algorithm 3), which we refer to as "**D**istributed **P**artially-observable Natural **P**olicy **G**radient" (DPPG).

4 Main Results with Discussion

In this section, we provide theoretical guarantee for Algorithms 2, 3, and 1.

4.1 Policy Evaluation

Several regularity conditions are necessary for a theoretical understanding of Algorithm 2, which are standard in optimization literature [7].

Definition 1 (Bounded gradient norm)**.** *There exists* $G \in \mathbb{R}_+$*, such that* $\forall \psi: \quad \mathbb{E}_{Z_h} \ \|g(\psi; Z_h)\|_2 \le G$

Definition 2 (Bounded gradient noise)**.** *There exists* $\sigma_g^2 > 0$*, such that* $\mathrm{Var}\left[g_h(\psi;\bar{Z}_h)\right] \le \sigma_g^2$

Definition 3 (Well-defined features)**.** $\exists$ *constant* $\mu_\phi \in \mathbb{R}_+$*, s.t.* $\mathbb{E}^{\pi^\theta}\left[\sum_{h=0}^{H-1} \gamma^h \ \phi_h(\bar{z}_h)\phi_h(\bar{z}_h)^\top\right] \succeq \mu_\phi \cdot \mathbb{I}_d$

Similar condition $\forall \theta : \mathbb{E}_{s\sim d^{\pi^\theta}}[\phi(s)\phi(s)^\top] \succeq \mu_\phi \cdot \mathbb{I}_d$ is commonly used in the theory of TD learning and Actor-critic algorithms for MDPs [6, 41, 45, 50]. We extend this to finite-horizon POMDPs by using multiple linear features ϕ_h to capture the temporal relations of POMDPs, and substituting the stationary distribution d^{π^θ} with the sample path probability. We also define $\nu_\phi = \mu_\phi(1-\gamma)$ as the ill-condition number of TD learning.

Next, we detail the error in distributed TD learning for POMDPs with linear function approximation. The proof can be found in Appendix B.2.

Theorem 1 (Bellman Error Bound for Algorithm 2)**.** *Set* $\iota_r = \frac{a}{\mu_\phi(1-\gamma)(r+a)}$ *where* a *is any constant larger than* $\frac{9}{2}$*. For any local learning rate* $\alpha_r \le \frac{\mu_\phi(1-\gamma)}{16T}$ *and global learning rate defined via* $\eta_r := \frac{\iota_r}{\alpha_r K}$*, the Bellman error of Algorithm 2's output parameter is controlled by the following upper-bound*

$$\mathbb{E}\mathcal{E}\left(\widehat{\psi}^R\right) \le \widetilde{O}\left(\frac{Hd_Q}{R^3} + \frac{1}{{\nu_\phi}^2}\left[\min\left\{\frac{1}{{\nu_\phi}^2 T^2}, \frac{1}{R^2}\right\} G^2 + \left(\min\left\{\frac{1}{{\nu_\phi}^2 K T^2}, \frac{1}{TR}\right\} + \frac{1}{MT}\right) H\sigma_g^2\right]\right) \tag{12}$$

if we do stochastic averaging with weights $w_r \propto (r+2)^2$ *after* $R_0 = \frac{128a}{{\nu_\phi}^2}$ *rounds of synchronizations.* σ_g^2 *and* G *are specified in Definitions 2 and 1.*

Theorem 1 immediately implies the following corollary, whose proof is provided in Appendix B.3.

Corollary 1 (Complexity of Algorithm 2)**.** *Pick* $K = \min\left\{\frac{T^{2/3}}{M^{1/3}}, \sqrt{\frac{2}{e}\frac{H\sigma_g^2}{G}\frac{T}{M}}, \frac{T}{M}\right\}$*. Under the conditions of Theorem 1, Algorithm 2 returns an* ϵ*-accurate Q-function estimate using only*

$$N_{\text{eval}} = \widetilde{O}\left(\frac{1}{M\epsilon}\left(\frac{H\sigma_g^2}{{\nu_\phi}^2} + Hd_Q\right)\right) \tag{13}$$

sample trajectories per agent, within $R = \max\left\{(MT)^{1/3}, \frac{e}{2}M, \sqrt{\frac{e}{2}\frac{G}{H\sigma_g^2}MT}\right\}$ *rounds of communications. Moreover, given sufficient computation* $\frac{T}{M} \geq \frac{1}{\nu_\phi{}^2}\frac{G^2}{H\sigma_g^2}$*, we can pick* $K = \frac{T^{2/3}}{M^{1/3}}$ *and obtain* $\widetilde{O}\left(\frac{H\sigma_g^2+Hd}{\nu_\phi{}^2 MT}\right)$ *sample complexity using only* $R = (MT)^{1/3}$ *rounds of communications. In either case, the algorithm requires* $C_{\text{eval}} = O(THd_Q)$ *basic operations per agent.*

Remark 2 (Interpretation of Theorem 1)*.* The coefficients $\min\left\{\frac{1}{\nu_\phi{}^2 KT^2}, \frac{1}{TR}\right\}$ and $\min\left\{\frac{1}{\nu_\phi{}^2 T^2}, \frac{1}{R^2}\right\}$ are novel ingredients compared with previous works such as [53, 55], owning to a precise characterization on the drift terms in Lemma 5. Corollary 1 shows that our algorithm achieves a linear speedup in sample complexity with only a few sublinear communication rounds relative to total computation T.

Remark 3 (Adjust Data Parallelism According to Feature Representation)*.* Corollary 1 reveals that, when our optimization conditions are mild (ν_ϕ is large and the gradients are small), we can safely adopt longer local updates $K = \frac{T^{2/3}}{M^{1/3}}$ to reduce communication costs while allowing computation to drive policy optimization. However, when the feature matrix becomes ill-defined (when μ_ϕ is small), we need to take fewer local steps to prevent the drift terms from exploding. A detailed analysis is provided in Lemma 5.

4.2 Policy Improvement

Next, we present regularity conditions for a theoretical understanding of Algorithm 3.

Definition 4 (Bounded KL Divergence)**.** *The KL divergence between the policies is finite. In particular, the optimal and the initial policy have a KL divergence below some finite D:* $D_{KL}\left(\pi_h^\star(\cdot|z_h)||\pi_h^{\theta_0}(\cdot|z_h)\right) < D$

Definition 5 (Smooth Score Functions)**.** $\exists \beta \in \mathbb{R}_+$*, s.t.* $\left|\nabla_\theta \ln \pi_h^{\theta'}(a_h|z_h) - \nabla_\theta \ln \pi_h^{\theta}(a_h|z_h)\right| \leq \beta \left\|\theta' - \theta\right\|_2$

Assumption 4 and 5, are essential even for the convergence of single-agent NPG with exact gradients. The following three conditions are unique to distributed NPG using noisy gradients, which are proposed to describe the statistical property of $\nabla \mathcal{L}_{\mathsf{A}}$.

Definition 6 (Bounded Gradient Noise)**.** *There exists* $\sigma_w^2 > 0$*, such that* $\mathbb{E}\left[\left\|\nabla_\omega L_{\mathsf{Q}} - \nabla_\omega \widehat{L}_{\mathsf{Q}}\right\|_2^2\right] \leq \sigma_w^2$

Definition 7 (Bounded Gradient Norm)**.** *The optimal and estimated NPG gradients are reasonably large:* $\exists W > 0$*,* $\left\|\omega_r^{k,m}\right\|_2$ *,* $\|\omega^\star\|_2 \leq W$ *a.s.*

Definition 8 (Non-degenerate Fisher Matrix)**.** *For any parameter* θ*, matrix* $\mathcal{F}_\theta$ *defined in Eq. (8) has positive eigenvalues greater than* $\frac{\mu_F}{2}$ *and less than* $\frac{L}{2}$*.*

Definition 8 implies that $\mathcal{L}_{\mathsf{A}}$ is μ_F-strongly convex, which ensures the uniqueness of $\omega^\star$. This is a common and not overly restrictive assumption in the theories of NPG algorithms for MDPs, from the original work proposing NPG [24] to more recent studies such as [57] (Eq. (37)) and [33] (Assumption 2.1). A broad class of practical policies, such as Gaussian policies, satisfy this condition (see page 5 of [33]).

Definition 9 (Sample-path Coverage)**.** *The concentratability coefficient between the trajectory probabilities induced by* $\pi^\star$ *and* π^{θ_r} *is almost surely finite:*

$$\kappa_r := \mathbb{E}^{\pi^\star}\frac{d\mathbb{P}^{\pi^\star}}{d\mathbb{P}^{\pi^{\theta_r}}} < +\infty \tag{14}$$

We also define $\kappa := \mathbb{E}_{\theta_{0:R-1}}\frac{1}{R}\sum_{r=0}^{R-1}\kappa_r$*, where the expectation is taken w.r.t. the randomness in NPG.*

Remark 4 (Interpretation of κ)*.* Assumption 9 is proposed to establish the global convergence of Algorithm 3. Eq. (14) is equivalent to $\kappa_r = \mathbb{E}^{\pi^\star}\prod_{t=0}^{H-1}\frac{\pi_t^\star(A_t|Z_t)}{\pi_t^{\theta_r}(A_t|Z_t)} < +\infty$, indicating the learned policies are exploratory enough to find the best actions. The assumption is satisfied or relaxed when π belongs to tabular soft-max or log-linear policies [2]. It generalizes similar restrictions from MDP theory [56], such as $\mathbb{E}_{(s,a)\sim d^{\pi^\star}}\left[\frac{d^{\pi^\star}(s,a)}{d^{\pi^{\theta_t}}(s,a)}\right] \leq C$, to POMDPs, however with the stationary distribution d replaced with sample path probability $\mathbb{P}$. A similar assumption is also adopted in POMDP theory [9] to establish global convergence. We provide an analysis on κ_r in Appendix C.1.2 using information theoretical technique.

We introduce two terms to evaluate the performance of solving the NPG direction ω. The first one is the statistical error, denoted as $\epsilon_{\text{stat}}(\omega;\theta_r) = \mathcal{L}_{\mathsf{Q}}(\omega;\theta_r;\pi^{\theta_r}) - \inf_\omega \mathcal{L}_{\mathsf{Q}}(\omega;\theta_r;\pi^{\theta_r})$, measures the accuracy of our linear regression solution. The second is the approximation error, $\epsilon_{\text{approx}}(\theta_r) = \inf_\omega \mathcal{L}_{\mathsf{Q}}(\omega;\theta_r;\pi^{\theta_r})$, represents the minimum error possible in approximating the Q-function with a linear combination of the coordinates of $\nabla_\theta f_\theta$. For a rich neural policy class, $\epsilon_{\text{approx}}(\theta_r)$ is very small [54]. In Section 3.1, we use the empirical estimate $\widehat{Q}$ in place of the Q-function for these definitions.

Building upon the literature on mirror descent [7], local SGD [46], and POMDP [9], we establish a global convergence rate for solving large POMDPs under softmax policy parameterization.

Theorem 2 (Global Convergence of Algorithm 3)**.** *Set* $\zeta_k = \frac{4}{\mu_F(k+a)}$ *where* $a = qI + 4$ *with* q *being a constant satisfying* $q\exp\left(-\frac{2}{q}\right) < \frac{L}{\mu_F}\sqrt{192\left(\frac{M+1}{M}\right)}$, $I = O\left(\frac{T^{1/3}}{M^{2/3}}\right)$, $\eta_r = \frac{\sqrt{2D/\beta}}{W\sqrt{R}}$. *Under regularity conditions 8, 5, and 9, the expected functional suboptimality of the outputs of Algorithm 3 is*

$$\mathsf{V}^{\pi^\star} - \mathbb{E}\left[\frac{1}{R}\sum_{r=0}^{R-1}\mathsf{V}^{\pi^{\theta_r}}\right] \lesssim HW\sqrt{\frac{\beta D}{R}} + \frac{1}{\mu_F}\sqrt{\frac{\kappa H L\sigma_w^2}{MK}} + \sqrt{\kappa H}\sqrt{\frac{1}{R}\sum_{r=0}^{R-1}\mathbb{E}\ \epsilon_{\text{approx}}(\theta_r)} \tag{15}$$

under exact policy evaluation.

The proof is derived from Lemma 1, 2 and 11.

Remark 5 (Interpretation of Theorem 2). The first term in Eq. (15) is determined by the initialization error D (cf. (4)). The second term stands for the optimization error in solving the NPG gradient ω using linear regression. It benefits from distributed training and is the source of a linear speedup in sample complexity. The third term, an error floor, accounts for function approximation error inherent in NPG methods. Compared to MDPs, the first term in the POMDP error bound scales with an extra $\sqrt{H}$ and the other terms scales with H, indicating increased complexity due to greater historical dependency. The last two terms also depend on κ (Eq. (14)), which is essential for achieving global convergence (cf. Remark 4).

The sample and communication complexity of Algorithm 3 by the following corollary, whose proof is provided in Appendix C.2.

Corollary 2 (Complexity of Algorithm 3)**.** *Pick* $K = \sqrt{\frac{T}{M}}$. *When the function approximation error is small, Algorithm 3 promises* ϵ*-optimality with*

$$N_{\text{improve}} = \widetilde{\mathcal{O}}\left(\frac{H^4W^4\beta^2D^2 + \kappa^2/\mu_F^4 H^2L^2\sigma_w^2}{M\epsilon^4}\right) \tag{16}$$

sample trajectories for each parallel machine. In total, we need to communicate

$$p_{\text{improve}} = O\left(M^{1/2}T^{1/2} + T^{5/6}M^{-1/6}\right)$$

times to synchronize the policy parameter θ_r *with its gradients* ω_r*, whose dimensions are* d_θ*. The computation consumption is* $C_{\text{improve}} = O\left(THd_\theta\right)$

Remark 6 (Speedup Effect). Eq. (16) implies that as parallel machines increase M times, the sample consumption per agent is reduced M-fold, showing a linear speedup in sample complexity. Meanwhile, the number of communications scales sublinearly with the computational budget and linearly with parameter dimension d, while irrelevant with the space sizes. This explains why we can significantly accelerate POMDP solvers when thousands of agents are simulated in parallel [42], even in continuous settings.

4.3 The Actor-critic Framework

Combining Theorem 2 and 1, we obtain a theoretical guarantee for our Actor-Critic algorithm 1. The proofs can be found in Appendix C.3.

Theorem 3. *Under the requirements of Theorem 2 and 1, running Algorithm 1 with T_e steps of policy evaluation ensures functional suboptimality of*

$$\mathsf{V}^{\pi^\star} - \mathbb{E}\left[\frac{1}{R}\sum_{r=0}^{R-1}\mathsf{V}^{\pi^{\theta_r}}\right] \lesssim \underbrace{HW\sqrt{\frac{\beta D}{R}}}_{\text{Initialization}} + \underbrace{\frac{1}{\mu_F}\sqrt{\frac{\kappa H L \sigma_w^2}{MK}}}_{\text{Optimization}} + \underbrace{\frac{1}{\nu_\phi}\sqrt{\frac{\kappa H^2\left(d_Q+\sigma_g^2\right)}{MT_e}}}_{\text{Evaluation}} + \underbrace{\sqrt{\kappa H \epsilon_{\text{app}}}}_{\text{Approximation}} \tag{17}$$

Where we abbreviated $\frac{1}{R}\sum_{r=1}^{R}\mathbb{E}\ \epsilon_{\text{approx}}(\theta_r)$ as ϵ_{app}. Regularity constants D, β, σ_w^2, κ, L, σ_g^2, and ν_ϕ are defined in Definitions 4, 5, 6, 9, 8, 2, and 3.

Corollary 3 (Complexity of Algorithm 1)**.**
Pick $K = \frac{T^{1/2}}{M^{1/2}}$ and $T_e = K$, Algorithm 1 achieves ϵ-optimality after consuming

$$N_{total} = N_{eval} + N_{improve} = \widetilde{O}\left(\frac{C^2}{M\epsilon^4}\right)$$

sample trajectories, where $C = H^2W^2\beta D + \frac{\kappa}{\mu_F^2}HL\sigma_g^2 + \kappa H^2\left(\frac{d_Q+\sigma_g^2}{\nu_\phi{}^2}\right)$. To achieve this, we need to send

$$P_{\text{total}} = O\left(\left(MT\right)^{1/2} + T^{5/6}M^{-1/6}\right)d_\theta + (MT)^{1/3}d_Q$$

floating-point parameters between the server and the clients. The total computation complexity is of order

$$C_{\text{total}} = O\left(TH(d_\theta + d_Q)\right)$$

Remark 7 (Benefit of Parallelism). Compared with Algorithm 3, Algorithm 1 suffers from an additional Bellman error, due to inexact policy evaluation. Fortunately, with linear and compatible function approximation, Algorithm 1 preserves the linear speedup in sample complexity and sublinear communication complexity, while its computation complexity is irrelevant with $\mathcal{S}, \mathcal{O}$. This explains why POMDP solvers benefit from massive parallel training.

Remark 8 (Complexity Trade-off). Our complexities do not go against the info-theoretical hardness of POMDPs [38]. Algorithm 1 only learns POMDPs whose value functions possess a linear structure, and the suboptimality suffers from an extra error floor due to approximation error. Optimizing POMDPs is still significantly harder than MDPs, which is revealed by additional dependency on $\sqrt{H}$ in the first term of Eq. (15) and a dependency on κ in the last three terms. This is because history-dependent policies have a significantly more complex geometry than MDPs [37].

5 Proof Outline

We outline the proofs for the major theorems in Section 4, and defer the details to Appendix C and B.

5.1 Policy Improvement

First, we extend the regret lemma in RL theory [2] to POMDPs with general policies. We provide a proof sketch in Appendix C.1.1 for completeness.

Lemma 1 (Regret Lemma for POMDPs, adapted from [9])**.**

$$\mathsf{V}^{\pi^\star} - \mathbb{E}\frac{1}{R}\sum_{r=0}^{R-1}\mathsf{V}^{\pi^{\theta_r}} \leq HW\sqrt{\frac{2\beta D}{R}} + \sqrt{\kappa H}\sqrt{\frac{1}{R}\sum_{r=0}^{R-1}\mathbb{E}\ \mathcal{L}_{\mathsf{A}}(\omega_r;\theta_r,\pi^{\theta_r})} \tag{18}$$

Next, we decompose $\mathcal{L}_{\mathsf{A}}$ with simpler terms, whose proof is provided in Appendix C.1.3.

Lemma 2 (Error Decomposition, [9])**.** $\mathcal{L}_{\mathsf{A}}(\omega;\theta,\pi^\theta) \leq 2\cdot\epsilon_{\text{stat}}(\omega;\theta) + 2\cdot\epsilon_{\text{approx}}(\theta) + 4\cdot\mathcal{E}^{\pi^\theta}\left(\widehat{\psi}\right)$.

Moreover, the statistical error enjoys linear speedup,

Fact 1 (Informal version of Lemma 11)**.** *Under careful design, $\mathbb{E}\ \epsilon_{\text{stat}}(\omega;\theta_r) \lesssim \frac{L}{\mu_F^2}\frac{\sigma_w^2}{MT}$*

Theorem 2 directly result from Lemmas 1, 2 and 11.

5.2 Policy Evaluation

To show Theorem 1, we introduce the potential function U to characterize the parameter's suboptimality.

Definition 10 (Potential Function in TD learning)**.** $U(\widetilde{\psi}^{r+1}) := \mathbb{E}\left\|\widetilde{\psi}^{r+1} - \psi^\star\right\|_2^2$

The improvement of the potential function decomposes into two terms:

$$U(\widetilde{\psi}^{r+1}) = U(\widetilde{\psi}^{r}) + \underbrace{2\mathbb{E}\left\langle \widetilde{\psi}^{r} - \psi^\star \ , \ \widetilde{\psi}^{r+1} - \widetilde{\psi}^{r} \right\rangle}_{\text{Progression Direction}} + \underbrace{\mathbb{E}\left\|\widetilde{\psi}^{r+1} - \widetilde{\psi}^{r}\right\|_2^2}_{\text{Progression Distance}} \tag{19}$$

The term "Progression Direction" refers to the angle between $\widetilde{\psi}^{r+1} - \widetilde{\psi}^{r}$ and $\widetilde{\psi}^{r} - \widetilde{\psi}^{\star}$, indicating the direction in which the synchronized parameters approach the global minimum $\psi^\star$. In contrast, "Progression Distance" measures the step size of the parameter update. These terms are governed by the following two lemmas, with their proofs available in the Appendix B.1.1 and B.1.2, respectively.

Lemma 3 (Progression Direction)**.** *For any constant $c \in \mathbb{R}_+$ the progression direction is controlled by*

$$\text{Progression Direction} \leq (\eta_r \alpha_r K) \cdot \left[\frac{1}{2c} U\left(\widetilde{\psi}^{r}\right) + \mathbb{E}\left\langle \widetilde{\psi}^{r} - \psi^\star, \mathbb{E}_{Z_H} g\left(\widetilde{\psi}^{r}; Z_H\right)\right\rangle + 8c \cdot \text{drift}(r)\right]$$

Lemma 4 (Progression Distance)**.** *The progression distance has the following upper bound:*

$$\text{Progression Distance} \leq (\eta_r \alpha_r K)^2 \cdot \left[4\left\|\mathbb{E}^{\pi}_{\bar{Z}} g\left(\widetilde{\psi}^{r}; \bar{Z}\right)\right\|_2^2 + \frac{2H\sigma_g^2}{MK} + 64 \cdot \text{drift}(r)\right]$$

The term $\iota_r := \eta_r \alpha_r K$ is the composite learning rate for each policy update, considering global and accumulated local updates. The constant c in Lemma 3 arises from Young's inequality (Fact 2), providing extra flexibility in designing the learning rate. Lemma 4 shows that periodic averaging reduces variance by a multiple of M, which is key to achieving linear speedup in sample complexity. In contrast to MDP theory, the noise variance scales with H, highlighting how the historical dependency in POMDPs accumulates sampling noise over the horizon. The term drift(r) measures how local models deviate from the recent global update.

Definition 11 (Local Drift)**.** *The local drift in round r is the average distance from the optimal parameter between consecutive local updates on each machine. drift$(r) := \frac{\sum_{m,k}}{MK}\mathbb{E}\left\|\psi^{r,k,m} - \bar{\psi}^{r}\right\|_2^2$*

We control the drift term with a sharp bound derived from Riemannian integrals (see Appendix B.1.3).

Lemma 5 (Local Drift Bound)**.** *When α^r is less than $\frac{\mu_\phi(1-\gamma)}{16T}$, the drift term in the r-th round is governed by the following upper bound,*

$$\text{drift}(r) \leq \Gamma(K; \nu_\phi)\left(\frac{H\sigma_g^2}{K} + 2G^2\right) \cdot \alpha_r^2$$

where $\Gamma(K;\nu_\phi)$ is defined as $\frac{e}{\nu_\phi{}^2}\left(1 - (1+\nu_\phi)e^{-\nu_\phi K}\right)$, which is dominated by $\frac{e}{(1-\gamma)^2\mu_\phi{}^2}$ when $\mu_\phi \neq 0$ and $\gamma \neq 1$, and it is controlled by $\frac{eK^2}{2}$ otherwise. We prove this lemma in Appendix B.1.3.

Lemma 5 directly reveals how a smaller local leaning rate helps suppress the local drift effect. We also propose a novel technique in Lemma 5, which connects the bound on the drift term with a Riemannian integral, allowing us to provide tight bound for the feature covariant matrix with varying regularity. If the problem is well-conditioned ($\nu_\phi > 0, \gamma < 1$), we can increase K without causing large drift, but when ill-conditioned ($\nu_\phi \to 0$), limiting local steps is necessary. Figure 2 illustrates these insights.

Bringing Lemma 5, 4, 4 to Eq. (19), we obtain the following Lemma, which reveals how the Bellman error relates to the potential function, learning rates, and the gradient noise.

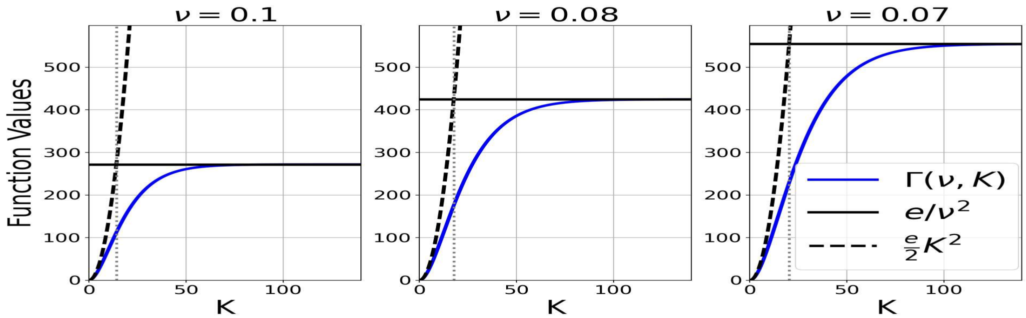

Figure 2: How local drift vary with K and ν_ϕ. When ν_ϕ is large, $\Gamma(K;\nu_\phi) \leq \frac{e}{{\nu_\phi}^2}$, we can take more local steps to save communication. As $\nu_\phi \to 0$, the drift effect quickly deteriorates, which necessitates frequent synchronization.

Lemma 6 (Error Decomposition of TD learning)**.** *When local learning rate satisfies $\alpha_r \leq \frac{\nu_\phi}{16T}$ while the composite learning rate obeys $\iota_r \leq \frac{\nu_\phi}{128}$, the Bellman error and the potential function obeys the following inequality:*

$$U\left(\widetilde{\psi}^{r+1}\right) \leq \left(1 - \frac{\nu_\phi}{2}\iota_r\right) U\left(\widetilde{\psi}^{r}\right) - \frac{\nu_\phi}{2}\mathbb{E}\,\mathcal{E}\left(\widetilde{\psi}^{r}\right)\iota_r + \frac{9}{4}\Gamma(K;\nu_\phi)\left(\frac{H\sigma_g^2}{RK^3} + \frac{2G^2}{RK^2}\right)\iota_r^2 + \left(\frac{2H\sigma_g^2}{MK}\right)\iota_r^2 \tag{20}$$

where $\Gamma(K;\nu_\phi)$ is defined in Lemmma 5. The proof is deferred to Appendix B.1.4.

We remark that the restrictions on α_r and ι_r are proposed to suppress the Bellman error. These constraints also meets the requirement of Lemma 5. Rearranging terms in Eq. (20) and sum up either sides with quadratically increasing weights after $r \geq R_0$, we conclude the proof of Theorem 1. The technique is shown in Appendix 1.

6 Conclusion and Future Work

This work establishes the first theoretical framework for understanding the empirical success of distributed policy optimization in solving large POMDPs. Comprehensive complexity results, novel analytical techniques, and extensive explanations are of independent interest to practitioners and theoreticians.

An exciting direction is to analyze how exploration helps improve regularity constants, which we leave for future work.

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A Preliminaries

A.1 Arithmetic Relations

Fact 2. ***Young's inequality***

- $\forall a, b \in \mathbb{R}, \eta \in \mathbb{R}_+ : ab \leq \frac{1}{2\eta}a^2 + \frac{\eta}{2}b^2$.
- $\forall A, a \in \mathbb{R}, \eta \in \mathbb{R}_+ : A^2 \leq (1+\eta)(A-a)^2 + \left(1 + \frac{1}{\eta}\right) a^2$
- $\forall a, b \in \mathbb{R} : (a+b)^2 \leq 2a^2 + 2b^2$

Fact 3. ***Jensen's inequality***

- $\forall n \in \mathbb{Z}_+, a_i \in \mathbb{R} : \left(\sum_{i=1}^n a_i\right)^2 \leq n \sum_{i=1}^n a_i^2$
- $\forall a, b, c, \in \mathbb{R} : (a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$

A.2 Policy Gradient for POMDPs

We extend the policy gradient theory from MDP to POMDPs. The results are adapted from [4, 9, 36, 58]. In partially observable reinforcement learning, we aim to find the policy that maximizes the value function

$$\mathsf{V}^\pi = \mathbb{E}^\pi \left[\sum_{h=0}^{H-1} \gamma^h \ r_h(S_h, A_h)\right]$$

We can write the Bellman equations in the following format:

$$\begin{aligned} \mathsf{V}_t^\pi(z_t) =& \mathbb{E}^{\pi_t}\left[\mathsf{Q}_t^\pi(\bar{z}_t = (z_t, A_t))\right] \\ \mathsf{Q}_t^\pi(z_t, a_t) =& \mathbb{E}\left[r_t(S_t, a_t) + \gamma \ \mathsf{V}_{t+1}^\pi(Z_{t+1} = (z_t, a_t, O_{t+1})) \middle| \bar{Z}_t = (z_t, a_t)\right] \end{aligned} \tag{21}$$

Similar to MDPs, the difference in the value functions of different policies is associated with the advantage function:

Lemma 7. (Performance Difference Lemma for POMDP) For any policy π', π,

$$\mathsf{V}^{\pi'} - \mathsf{V}^\pi = \mathbb{E}^{\pi'}\left[\sum_{h=0}^{H-1} \gamma^h \ \mathsf{A}_h^\pi(Z_h, A_h)\right]$$

We can also smoothly generalize policy gradient theorems [47] to POMDPs by the following lemma:

Theorem 4 (Policy Gradient Theorem for Finite-horizon Discounted Reward POMDPs)**.**

$$\nabla_\theta \mathsf{V}^\pi = \mathbb{E}^{\pi^\theta}\left[\sum_{h=0}^{H-1} \gamma^h \mathsf{Q}_t^{\pi^\theta}(Z_t, A_t) \nabla_\theta \ln \pi_t^\theta(A_t|Z_t)\right] = \mathbb{E}^{\pi^\theta}\left[\sum_{h=0}^{H-1} \gamma^h \mathsf{A}_t^{\pi^\theta}(Z_t, A_t) \nabla_\theta \ln \pi_t^\theta(A_t|Z_t)\right] \tag{22}$$

Remark 9. We can write Eq. (22) in another way:

$$\nabla_\theta \mathsf{V}^{\pi^\theta} = \mathbb{E}_{\mathcal{P}}^{\pi^\theta}\left[\sum_{h=0}^{H-1} \gamma^h \frac{\nabla_\theta \pi_h^\theta(A_h|Z_h)}{\pi_h^\theta(A_h|Z_h)} \mathsf{A}_h^{\pi^\theta}(Z_h, A_h)\right] \tag{23}$$

which sheds light on how to approximate the value function by simpler structures:

Corollary 4 (Importance Sampling). *For finite horizon, discounted reward POMDP, the following function*

$$\widetilde{L}(\theta;\theta_t) := \mathsf{V}^{\pi^{\theta_t}} + \mathbb{E}^{\pi^{\theta_t}}\left[\sum_{h=0}^{H-1}\gamma^h \frac{\pi_h^{\theta}(A_h|Z_h)}{\pi_h^{\theta_t}(A_h|Z_h)}\mathsf{A}_h^{\pi^{\theta_t}}(Z_h,A_h)\right]$$

satisfies

$$\text{(i)}\quad \widetilde{L}(\theta;\theta_t)\big|_{\theta=\theta_t} = \mathsf{V}^{\pi^{\theta}}\big|_{\theta=\theta_t} \qquad \text{(ii)}\quad \nabla_\theta\ \widetilde{L}(\theta;\theta_t)\big|_{\theta=\theta_t} = \nabla_\theta \mathsf{V}^{\pi^{\theta}}\big|_{\theta=\theta_t}$$

which implies that it can be used as a first-order approximator of the value function $\mathsf{V}^{\pi^{\theta}}$ in the neighborhood of θ_t:

$$\forall\epsilon>0,\ \exists\delta>0,\ s.t.\ \forall\theta : \|\theta-\theta_t\|\le\delta,\qquad \left|\mathsf{V}^{\pi^{\theta}} - \widetilde{L}(\theta;\theta_t)\right|\le\epsilon$$

This corollary illuminates how Theorem 4 relates with TRPO [43] and PPO [44] algorithms.

B Distributed TD learning for POMDPs

B.1 Proof of Auxiliary Lemmas

To allow for more flexibility, we extend the definition of Bellman error in Eq. (4) to any two stacked parameters:

Definition 12 (Extended Bellman Error)**.** *Given two parameters ψ^1 and ψ^2. Let $\widehat{\mathsf{Q}}^i$ denote the Q-function estimate constructed from $\widehat{\mathsf{Q}}_h^i = \langle\phi_h(\bar{Z}_h),\psi_h^i\rangle$ for $i\in\{1,2\}$. Define*

$$\mathcal{E}^\pi(\psi_1,\psi_2) := \mathbb{E}^\pi\left[\sum_{h=0}^{H-1}\gamma^h\frac{\left(\widehat{\mathsf{Q}}_h^1(\bar{Z}_h)-\widehat{\mathsf{Q}}_h^2(\bar{Z}_H)\right)^2}{2}\right] = (\psi_1-\psi_2)^\top\,\mathbb{E}\left[\sum_{h=0}^{H-1}\gamma^h\ \phi_h(\bar{z}_h)^\top\phi_h(\bar{z}_h)\right](\psi_1-\psi_2)$$

It is straightforward to see that $\mathcal{E}^\pi(\psi,\psi^\star)=\mathcal{E}^\pi(\psi)$.

We also need the feature vectors to satisfy several regularity conditions to support our theoretical analysis. These conditions are standard in the theory of linear function approximation [5, 11, 50]. Without loss of generality, we require the feature vectors to be normalized 2-norms, so that the norm of the parameters and the Bellman error defined in Eq. (4)) have the following (crude) upper bounds

$$\|\psi\|_2 \le \sqrt{Hd}\ ,\quad \mathcal{E}(\psi)\le\|\psi-\psi^\star\|_2 \tag{24}$$

Readers may refer to Lemma B.2 of [13] or Section 2.1 in [22] for a reference. On the other hand, condition 3 ensure that

$$\|\psi-\psi^\star\|_2^2 \le \frac{1}{\mu_\phi}\ \mathcal{E}(\psi),\quad \|\psi_1-\psi_2\|_2^2\le\frac{1}{\mu_\phi}\ \mathcal{E}^\pi(\psi_1,\psi_2) \tag{25}$$

Comparing Eq. (24) with Eq.(25), we discover the relationship between the potential function and the Bellman error:

$$\mu_\phi\cdot U\left(\widetilde{\psi}^r\right)\le\mathbb{E}\,\mathcal{E}\left(\widetilde{\psi}^r\right)\le U\left(\widetilde{\psi}^r\right) \tag{26}$$

B.1.1 Proof of Lemma 3

Proof. In Lemma 3, we aim to show that for any constant $c\in\mathbb{R}_+$:

$$\underbrace{2\cdot\mathbb{E}\left\langle\widetilde{\psi}^r-\psi^\star\ ,\ \widetilde{\psi}^{r+1}-\widetilde{\psi}^r\right\rangle}_{\text{Progression Direction}}$$

$$\le\eta^r\alpha^r K\cdot\left[-2(1-\gamma)\ \underbrace{\mathbb{E}\,\mathcal{E}(\widetilde{\psi}^r)}_{\text{Bellman error}}\ +\frac{1}{c}\cdot\underbrace{\mathbb{E}\left\|\widetilde{\psi}^r-\psi^\star\right\|_2^2}_{\text{Potential function}}+16c\cdot\underbrace{\frac{\sum_{m,k}}{MK}\mathbb{E}\left\|\psi^{r,k,m}-\widetilde{\psi}^r\right\|_2^2}_{\text{Local drift}}\right]$$

We will heavily rely on the independence relation of successive rollout trajectories. For a precise characterization, we introduce a set of sigma algebras to describe the relationship between several random variables.

Definition 13. *Define the random events occurring before the derivation of $\psi_h^{r,k,m}$ as*

$$\mathcal{F}_h^{r,k,m} = \sigma\left(\psi^{r,k-1,m}, \bar{z}_{h+1}^{r,k-1,m}\right) = \sigma\left(\bar{\psi}^r, \{\bar{z}_{h+1}^{r,t,m}\}_{t=0}^{k-1}\right) \qquad \mathcal{F}^{r,k,m} := \bigcup_{h=0}^{H-1} \mathcal{F}_h^{r,k,m}$$

We further define the following sigma-algebras to characterize the randomness before we update other parameters:

$$\mathcal{F}_h^{r,k} := \bigcup_{m=0}^{M-1} \mathcal{F}_h^{r,k,m} \qquad \mathcal{F}^{r,k} := \bigcup_{h=0}^{H-1} \mathcal{F}_h^{r,k} \qquad \mathcal{F}_h^{r} := \bigcup_{k=0}^{K} \mathcal{F}_h^{r,k} \qquad \mathcal{F}^{r} := \bigcup_{h=0}^{H-1} \mathcal{F}_h^{r}$$

To prove Lemma 3, we unroll the update rule to exress the parameters by sample of the semi-gradients, before we telescope the sample gradients with their expectations and invoke their continuity conditions. First, we notice that from Algorithm 2,

$$\widetilde{\psi}^{r+1} = (1-\eta_r)\,\widetilde{\psi}^r + \eta_r \bar{\psi}^r = \widetilde{\psi}^r - \eta_r \alpha_r \cdot \frac{\sum_m}{M} \sum_{k=0}^{K-1} g_h(\psi^{r,k,m}; \bar{z}_{h+1}^{r,k,m}) \tag{27}$$

Eq. (27) suggests that the following relation holds for any realization of $\bar{\psi}^r, \bar{\psi}^{r+1}$ and $\psi^{r,k,m}$:

$$\begin{aligned}
&2\left\langle \widetilde{\psi}^r - \psi^\star \,,\, \widetilde{\psi}^{r+1} - \widetilde{\psi}^r \right\rangle \\
=& -2\,(\eta^r\alpha^r) \sum_{h=0}^{H-1} \left\langle \widetilde{\psi}_h^r - \psi_h^\star, \frac{1}{M} \sum_{m=0}^{M-1}\sum_{k=0}^{K-1} g_h\left(\psi^{r,k,m}; \bar{z}_{h+1}^{r,k,m}\right) - \mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}} \; g_h\left(\psi^{r,k,m}; \bar{Z}_{h+1}^{r,k,m}\right) \right\rangle \\
&+ 2\,(\eta^r\alpha^r)\,\frac{1}{M} \sum_{m=0}^{M-1}\sum_{h=0}^{H-1}\sum_{k=0}^{K-1} \left\langle \widetilde{\psi}_h^r - \psi_h^\star, \mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}} \; g_h\left(\psi^{r,k,m}; \bar{Z}_{h+1}^{r,k,m}\right) \right\rangle
\end{aligned}$$

By the towering rule, we see that the telescoped term has zero mean,

$$\begin{aligned}
&\mathbb{E}\,\left\langle \widetilde{\psi}_h^r - \psi_h^\star \,,\, g_h\left(\psi^{r,k,m}; \bar{z}_{h+1}^{r,k,m}\right) - \mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}} \; g_h\left(\psi^{r,k,m}; \bar{Z}_{h+1}^{r,k,m}\right) \right\rangle \\
=&\mathbb{E}\,\left[\left\langle \widetilde{\psi}_h^r - \psi_h^\star, \mathbb{E}\,\left[g_h\left(\psi^{r,k,m}; \bar{z}_{h+1}^{r,k,m}\right) - \mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}} \; g_h\left(\psi^{r,k,m}; \bar{Z}_{h+1}^{r,k,m}\right)\right\rangle \Big| \mathcal{F}^{r,k,m}\right]\right] = 0
\end{aligned}$$

which implies that the left-hand-side in Lemma 3 equals

$$\begin{aligned}
&2\cdot\mathbb{E}\left\langle \widetilde{\psi}^r - \psi^\star \,,\, \widetilde{\psi}^{r+1} - \widetilde{\psi}^r \right\rangle \\
=&2\,(\eta^r\alpha^r)\cdot\mathbb{E}\left\langle \widetilde{\psi}^r - \psi^\star \,,\, \frac{1}{M}\sum_{m=0}^{M-1}\sum_{k=0}^{K-1} \mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}} \; g\left(\psi^{r,k,m}; \bar{z}_{h+1}^{r,k,m}\right) \right\rangle \\
=&2\,(\eta^r\alpha^r)\cdot\sum_{k=0}^{K-1}\mathbb{E}\left\langle \widetilde{\psi}^r - \psi^\star \,,\, \mathbb{E}_Z \; g\left(\widetilde{\psi}^r; Z\right) \right\rangle \\
&+ 2\,(\eta^r\alpha^r)\cdot\sum_{k=0}^{K-1}\mathbb{E}\left\langle \widetilde{\psi}^r - \psi^\star \,,\, \frac{1}{M}\sum_{m=0}^{M-1} \mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}} \; g\left(\psi^{r,k,m}; \bar{z}_{h+1}^{r,k,m}\right) - \mathbb{E}_Z \; g\left(\widetilde{\psi}^r; Z\right) \right\rangle
\end{aligned}$$

For the first term, Assumption 1 implies that

$$\begin{aligned}
&2\,(\eta^r\alpha^r)\cdot\sum_{k=0}^{K-1}\mathbb{E}\left\langle \widetilde{\psi}^r - \psi^\star \,,\, \mathbb{E}_Z \; g\left(\widetilde{\psi}^r; Z\right) \right\rangle \\
\le& 2\eta^r\alpha^r K\cdot\left(\frac{\mathbb{E}\,\left\|\widetilde{\psi}^r - \psi^\star\right\|_2^2}{2} + \frac{\left\|\mathbb{E}_Z g\left(\widetilde{\psi}^r; Z\right)\right\|_2^2}{2}\right) \le 2\eta^r\alpha^r K\cdot\left(\frac{\mathbb{E}\,\left\|\widetilde{\psi}^r - \psi^\star\right\|_2^2}{2} + \frac{G^2}{2}\right)
\end{aligned}$$

For the second term, Lemma 9 implies

$$
\begin{aligned}
&2\left(\eta^r\alpha^r\right)\cdot\sum_{k=0}^{K-1}\mathbb{E}\left\langle\widetilde{\psi}^r-\psi^\star\ ,\ \frac{1}{M}\sum_{m=0}^{M-1}\mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}}\ g\left(\psi^{r,k,m};\bar{z}_{h+1}^{r,k,m}\right)-\mathbb{E}_Z\ g\left(\widetilde{\psi}^r;Z\right)\right\rangle\\
\leq&2\eta^r\alpha^r\sum_{k=0}^{K-1}\mathbb{E}\ \left(\frac{\left\|\widetilde{\psi}^r-\psi^\star\right\|}{2c}+\frac{c}{2}\frac{\sum_{m=0}^{M-1}}{M}\left\|\mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}}\ g\left(\psi^{r,k,m};\bar{z}_{h+1}^{r,k,m}\right)-\mathbb{E}_Z\ g\left(\widetilde{\psi}^r;Z\right)\right\|_2^2\right)\\
\leq&2\eta^r\alpha^r\sum_{k=0}^{K-1}\mathbb{E}\ \left(\frac{\left\|\widetilde{\psi}-\psi^\star\right\|_2^2}{2c}+\frac{c}{2}\frac{\sum_{m=0}^{M-1}}{M}16\left\|\psi^{r,k,m}-\widetilde{\psi}^r\right\|_2^2\right)
\end{aligned}
$$

which results in

$$
\underbrace{2\cdot\mathbb{E}\left\langle\widetilde{\psi}^r-\psi^\star\ ,\ \widetilde{\psi}^{r+1}-\widetilde{\psi}^r\right\rangle}_{\text{Progression Direction}}\leq\eta^r\alpha^rK\left[\left(1+\frac{1}{c}\right)\mathbb{E}\ \left\|\widetilde{\psi}^r-\psi^\star\right\|_2^2+G^2+16c\cdot\underbrace{\frac{\sum_{m,k}}{MK}\mathbb{E}\left\|\psi^{r,k,m}-\widetilde{\psi}^r\right\|_2^2}_{\text{Local drift}}\right]
$$

□

Before we proceed to prove the next lemma (Lemma 4), we need to characterize how the variance in sample gradients is reduced by sampling independent rollout trajectories from the same policy.

Lemma 8 (Variance Reduction)**.**

$$
\mathbb{E}\left\|\frac{1}{MK}\sum_{m=0}^{M}\sum_{k=0}^{K-1}g_h(\psi^{r,k,m};\bar{z}_{h+1}^{r,k,m})-\mathbb{E}^{\pi}_{\bar{Z}_{h+1}^{r,k,m}}\ g_h(\psi^{r,k,m};\bar{Z}_{h+1}^{r,k,m})\right\|_2^2\leq\frac{1}{MK}\sigma_g^2
$$

This result is a smooth generalization of Lemma 7 in [53] and we omit the proof for brevity.

We also need to describe the continuity of the semi-gradients, which is summarized in the following lemma.

Lemma 9 (Lipchitzness of the Expected Semi-gradient)**.** *The ensemble mean of the stacked semi-gradient has Lipschitz constant 4:* $\left\|\mathbb{E}_{\bar{Z}_{h+1}}\ g(\psi^1;\bar{Z}_{h+1})-\mathbb{E}_{\bar{Z}_{h+1}}\ g(\psi^2;\bar{Z}_{h+1})\right\|_2\leq 4\left\|\psi^1-\psi^2\right\|_2$

Proof. Under linear representation, the ensemble mean of the semi-gradient at each step takes the following form:

$$
\begin{aligned}
&\mathbb{E}_{\bar{Z}_{h+1}}\ g_h(\psi^1;\bar{Z}_{h+1})\\
=&-\gamma^h\mathbb{E}\ \left[\phi_h(\bar{Z}_H)(r(\bar{Z}_H)+\gamma\phi_{h+1}(\bar{Z}_{h+1})^\top\psi_{h+1}^1-\phi_h(\bar{Z}_H)^\top\psi_h^1)\right]\\
=&-\gamma^h\mathbb{E}\ \left[\phi_h(\bar{Z}_H)r(\bar{Z}_H)\right]-\gamma^{h+1}\mathbb{E}\left[\phi_h(\bar{Z}_H)\phi_{h+1}(\bar{Z}_{h+1})\right]^\top\psi_{h+1}^1-\gamma^h\mathbb{E}\phi_h(\bar{Z}_H)\phi_h(\bar{Z}_H)^\top\psi_h^1
\end{aligned}
$$

Triangle inequality and Cauchy-Schwartz inequality suggest that

$$
\begin{aligned}
&\left\|\mathbb{E}_{\bar{Z}_{h+1}}\ g_h(\psi^1;\bar{Z}_{h+1})-\mathbb{E}_{\bar{Z}_{h+1}}\ g_h(\psi^2;\bar{Z}_{h+1})\right\|_2\\
=&\left\|-\gamma^{h+1}\mathbb{E}\left[\phi_h(\bar{Z}_H)\phi_{h+1}(\bar{Z}_{h+1})\right]^\top\left(\psi_{h+1}^1-\psi_{h+1}^2\right)-\gamma^h\mathbb{E}\phi_h(\bar{Z}_H)\phi_h(\bar{Z}_H)^\top\left(\psi_h^1-\psi_h^2\right)\right\|_2\\
=&\gamma^{h+1}\left\|\mathbb{E}\left[\phi_h(\bar{Z}_H)\phi_{h+1}(\bar{Z}_{h+1})\right]^\top\left(\psi_{h+1}^1-\psi_{h+1}^2\right)\right\|_2+\gamma^h\left\|\mathbb{E}\phi_h(\bar{Z}_H)\phi_h(\bar{Z}_H)^\top\left(\psi_h^1-\psi_h^2\right)\right\|_2\\
=&\gamma^{h+1}\left\|\psi_{h+1}^1-\psi_{h+1}^2\right\|_2+\gamma^h\left\|\psi_h^1-\psi_h^2\right\|_2
\end{aligned}
$$

which implies

$$
\left\|\mathbb{E}_{\bar{Z}_{h+1}}\ g_h(\psi^1;\bar{Z}_{h+1})-\mathbb{E}_{\bar{Z}_{h+1}}\ g_h(\psi^2;\bar{Z}_{h+1})\right\|_2^2\leq 2\gamma^{2h+2}\left\|\psi_{h+1}^1-\psi_{h+1}^2\right\|_2^2+2\gamma^{2h}\left\|\psi_h^1-\psi_h^2\right\|_2^2
$$

Stacking semi-gradients according to Eq. (6), we conclude

$$\left\|\mathbb{E}_{\bar{Z}_H}\ g(\psi^1;\bar{Z}_H)-\mathbb{E}_{\bar{Z}_{h+1}}\ g(\psi^2;\bar{Z}_{h+1})\right\|_2^2=\sum_{h=0}^{H-1}\left\|\mathbb{E}_{\bar{Z}_{h+1}}\ g_h(\psi^1;\bar{Z}_{h+1})-\mathbb{E}_{\bar{Z}_{h+1}}\ g_h(\psi^2;\bar{Z}_{h+1})\right\|_2^2$$
$$\leq\sum_{h=0}^{H-1}2\gamma^{2h+2}\left\|\psi^1_{h+1}-\psi^2_{h+1}\right\|_2^2+2\gamma^{2h}\left\|\psi^1_h-\psi^2_h\right\|_2^2\leq 4\sum_{h=0}^{H-1}\left\|\psi^1_h-\psi^2_h\right\|_2^2=4\left\|\psi^1-\psi^2\right\|_2^2$$

which is what we desired. □

Building upon Lemma 8 and 9, we are ready to show Lemma 4.

B.1.2 Proof of Lemma 4

In Lemma 4 we claim that

$$\underbrace{\mathbb{E}\ \left\|\widetilde{\psi}^{r+1}-\widetilde{\psi}^r\right\|_2^2}_{\text{Progression Distance}}\leq(\eta^r\alpha^r K)^2\cdot\left[4\left\|\mathbb{E}^{\pi}_{\bar{Z}}g\left(\widetilde{\psi}^r;\bar{Z}\right)\right\|_2^2+64\underbrace{\frac{\sum_{m,k}}{MK}\mathbb{E}\left[\left\|\psi^{r,k,m}-\widetilde{\psi}^r\right\|_2^2\right]}_{\text{Local drift}}+2\underbrace{\frac{H\sigma_g^2}{MK}}_{\text{Variance}}\right]$$

Proof. Similar to the proofs in Appendix B.1.1, the update rule in Eq. (27) and Young's inequality suggests that the progression distance decomposes into two terms:

$$\begin{aligned}\mathbb{E}\ \left\|\widetilde{\psi}^{r+1}-\widetilde{\psi}^r\right\|_2^2&=(\eta^r\alpha^r)^2\cdot\sum_{h=0}^{H-1}\mathbb{E}\ \left\|\frac{1}{M}\sum_{m=0}^{M}\sum_{k=0}^{K-1}g_h(\psi^{r,k,m};z^{r,k,m}_{h+1})\right\|_2^2\\&\leq\underbrace{2\left(\eta^r\alpha^r\right)^2\sum_{h=0}^{H-1}\mathbb{E}\left\|\frac{1}{M}\sum_{m=0}^{M}\sum_{k=0}^{K-1}g_h(\psi^{r,k,m};z^{r,k,m}_{h+1})-\mathbb{E}^{\pi}_{\bar{Z}^{r,k,m}_{h+1}}\ g_h(\psi^{r,k,m};\bar{Z}^{r,k,m}_{h+1})\right\|_2^2}_{\text{Variance}}\\&\quad\underbrace{+\,2\left(\eta^r\alpha^r\right)^2\sum_{h=0}^{H-1}\mathbb{E}\left\|\frac{1}{M}\sum_{m=0}^{M}\sum_{k=0}^{K-1}\mathbb{E}^{\pi}_{\bar{Z}^{r,k,m}_{h+1}}\ g_h(\psi^{r,k,m};\bar{Z}^{r,k,m}_{h+1})\right\|_2^2}_{\text{Bias}}\end{aligned}\tag{28}$$

The first term is associated with the noise in the sampling process, while the second term is related to the local drift defined in 11 and the bias in the semi-gradient.

Lemma 8 indicates that the variance term is governed by the following bound

$$\text{Variance}\leq 2\left(\eta^r\alpha^r\right)^2\sum_{h=0}^{H-1}\frac{K\sigma_g^2}{M}=2(\eta^r\alpha^r)^2K\frac{H\sigma_g^2}{M}\tag{29}$$

The bias term is related to the Bellman error. Specifically, Jensen's inequality suggests that

$$\sum_{h=0}^{H-1}\mathbb{E}\left\|\frac{1}{M}\sum_{m=0}^{M}\sum_{k=0}^{K-1}\mathbb{E}^{\pi}_{\bar{Z}^{r,k,m}_{h+1}}\ g_h(\psi^{r,k,m};\bar{Z}^{r,k,m}_{h+1})\right\|_2^2$$
$$\leq\sum_{h=0}^{H-1}K^2\cdot\frac{\sum_{m,k}}{MK}\mathbb{E}\left[\mathbb{E}\left[\left\|\mathbb{E}^{\pi}_{\bar{Z}^{r,k,m}_{h+1}}\ g_h(\psi^{r,k,m};\bar{Z}^{r,k,m}_{h+1})-\mathbb{E}^{\pi}_{\bar{Z}}\ g_h(\widetilde{\psi}^r;\bar{Z})+\mathbb{E}^{\pi}_{\bar{Z}}\ g_h(\widetilde{\psi}^r;\bar{Z})\right\|_2^2\middle|\mathcal{F}^{r,k,m}\right]\right]$$

With a telescoping trick, we observe that

$$\sum_{h=0}^{H-1} \mathbb{E} \left\| \frac{1}{M} \sum_{m=0}^{M} \sum_{k=0}^{K-1} \mathbb{E}^{\pi}_{\bar{Z}^{r,k,m}_{h+1}} \, g_h(\psi^{r,k,m}; \bar{Z}^{r,k,m}_{h+1}) \right\|_2^2$$
$$\leq 2K^2 \cdot \frac{\sum_{m,k}}{MK} \left(\mathbb{E}\left[\mathbb{E}\left[\left\| \mathbb{E}^{\pi}_{\bar{Z}^{r,k,m}_{h+1}} \, g(\psi^{r,k,m}; \bar{Z}^{r,k,m}_{h+1}) - \mathbb{E}^{\pi}_{\bar{Z}} \, g(\widetilde{\psi}^r; \bar{Z}) \right\|_2^2 \right] \middle| \mathcal{F}^{r,k,m} \right] + \sum_{h=0}^{H-1} \left\| \mathbb{E}^{\pi} g_h(\widetilde{\psi}^r; \bar{Z}) \right\|_2^2 \right) \quad (30)$$
$$\leq 32K^2 \cdot \frac{\sum_{m,k}}{MK} \mathbb{E}\left[\left\| \psi^{r,k,m} - \widetilde{\psi}^r \right\|_2^2 \right] + 2K^2 \sum_{h=0}^{H-1} \left\| \mathbb{E}^{\pi} g_h(\widetilde{\psi}^r; \bar{Z}) \right\|_2^2$$

where the first step is by Young's inequality (Fact 2) and the second is due to the Lipchitzness of the average semi-gradients (Lemma 9). The proof is completed after we bring Eq. (30), (29) to Eq. (28). □

In what follows, we will show how the upper bound on the local-drift term scales with the gradient variance and the learning rate (Lemma 5). As a preparation work, we need to relate the semi-gradients with the Bellman error.

Lemma 10 (First-order Relation for Extended Bellman Error)**.**

$$\mathbb{E} \left\langle \psi^1 - \psi^2 \ , \ \mathbb{E}_{\bar{Z}} \left[g(\psi^1; \bar{Z}) \right] - \mathbb{E}_{\bar{Z}} \left[g(\psi^2; \bar{Z}) \right] \right\rangle \leq -(1-\gamma) \mathbb{E} \ \mathcal{E}(\psi^1; \psi^2)$$

Proof.

$$g_h(\psi^1; \bar{z}) = \gamma^h \phi_h \left(r + \gamma \phi_{h+1}^\top \psi_{h+1}^1 - \psi_h^\top \psi_h^1 \right) g_h(\psi^2; \bar{z}) = \quad \gamma^h \phi_h \left(r + \gamma \phi_{h+1}^\top \psi_{h+1}^2 - \psi_h^\top \psi_h^2 \right)$$

So we have

$$LHS = \mathbb{E} \sum_{h=0}^{H-1} \langle \psi_h^1 - \psi_h^2, g_h(\psi_1) - g_h(\psi_2) \rangle = -\mathbb{E} \sum_{h=0}^{H-1} \langle \psi_h^1 - \psi_h^2, -\gamma^h \phi_h (\gamma \phi_{h+1}^\top (\psi_{h+1}^1 - \psi_{h+1}^2)) - \phi_h^\top (\psi_h^1 - \psi_h^2) \rangle$$
$$= -\mathbb{E} \sum_{h=0}^{H-1} \gamma^{h+1} \langle \phi_h, \psi_h^1 - \psi_h^2 \rangle \cdot \langle \phi_{h+1}, \psi_{h+1}^1 - \psi_{h+1}^2 \rangle + \gamma^h \langle \phi_h, \psi_h^1 - \psi_h^2 \rangle$$
$$= \mathbb{E} - \gamma \sum_{h=0}^{H-1} \gamma^h u_h u_{h+1} + 2\mathcal{E}(\psi_1, \psi_2) \leq \gamma \mathbb{E} \sum_{h=0}^{H-1} \gamma^h \left(\frac{u_h^2}{2\gamma} + \frac{\gamma u_h^2}{2} \right) - 2\mathcal{E}(\psi_1, \psi_2)$$
$$= -\mathcal{E}\left(\psi^1, \psi^2\right) + \gamma(\mathcal{E}\left(\psi^1, \psi^2\right) - \frac{u_0^2}{2}) + 2\mathcal{E}(\psi_1, \psi_2) \leq -(1-\gamma)\mathcal{E}(\psi_1, \psi_2) = RHS$$

We remark that we write $\phi_t(\bar{z}_t)^\top (\psi_t^k - \psi^\star)$ as u_t for brevity. □

Now we are ready to prove Lemma 5.

B.1.3 Proof of Lemma 5

We would like to show that

$$\underbrace{\frac{\sum_{m,k}}{MK} \mathbb{E}\left[\left\| \psi^{r,k,m} - \widetilde{\psi}^r \right\|_2^2 \right]}_{\text{Local drift}} \leq (\alpha^r)^2 \cdot \Gamma(K; \nu_\phi) \cdot \left(\frac{H\sigma_g^2}{K} + 2G^2 \right)$$

where

$$\Gamma(K; \nu_\phi) := e \cdot \frac{1 - (1 + \nu_\phi K) e^{-\nu_\phi K}}{{\nu_\phi}^2} \quad (31)$$

$\Gamma(K; \nu_\phi)$ approaches $\frac{eK^2}{2}$ when $\gamma = 1$ or $\mu_\phi = 0$, while is dominated by $\frac{e}{{\mu_\phi}^2 (1-\gamma)^2}$ when $0 < \alpha^r < \frac{\mu_\phi (1-\gamma)}{16}$. This result directly implies that,

- When $\mu_\phi \neq 0$ and $\gamma \neq 1$ and the learning rate satisfies $\alpha^r < \frac{\mu_\phi(1-\gamma)}{16}$,

$$\underbrace{\frac{\sum_{m,k}}{MK}\mathbb{E}\left[\left\|\psi^{r,k,m} - \widetilde{\psi}^r\right\|_2^2\right]}_{\text{Local drift}} \leq \frac{e \cdot \alpha_r^2}{(1-\gamma)^2 {\mu_\phi}^2}\left(\frac{H\sigma_g^2}{K} + 2G^2\right)$$

- When $\mu_\phi = 0$ or $\gamma = 1$ and the learning rate satisfies $\alpha^r < \frac{\mu_\phi(1-\gamma)}{16}$,

$$\underbrace{\frac{\sum_{m,k}}{MK}\mathbb{E}\left[\left\|\psi^{r,k,m} - \widetilde{\psi}^r\right\|_2^2\right]}_{\text{Local drift}} \leq \frac{e \cdot (\alpha^r)^2 K^2}{2}\left(\frac{H\sigma_g^2}{K} + 2G^2\right)$$

which is the cornerstone of Theorem 1.

Proof. The proof is a stronger version of proof of Lemma 9 in [53]. We express the parameter ψ in terms of the sampled semi-gradients, then we use a telescoping trick to replace the sampled gradients with their expected value.

$$\begin{aligned}
&\underbrace{\mathbb{E}\left[\left\|\psi^{r,k,m} - \widetilde{\psi}^r\right\|_2^2\right]}_{\text{Local drift}} \\
=&\sum_h \mathbb{E}\left[\left\|\psi_h^{r,k-1,m} - \widetilde{\psi}_h^r + \psi_h^{r,k,m} - \psi_h^{r,k-1,m}\right\|_2^2\right] \\
=&\sum_h \mathbb{E}\left[\left\|\psi_h^{r,k-1,m} - \widetilde{\psi}_h^r + \alpha^r g_h(\psi^{r,k-1,m}; \bar{z}_{h+1}^{r,k-1,m})\right\|_2^2\right] \\
=&\sum_h \mathbb{E}\left[\left\|\psi_h^{r,k-1,m} - \widetilde{\psi}_h^r + \alpha^r \mathbb{E}_{\bar{Z}_{h+1}^{r,k-1,m}}\ g_h(\psi^{r,k-1,m}; \bar{z}_{h+1}^{r,k-1,m})\right\|_2^2\right] \\
&+ (\alpha^r)^2 \cdot \mathbb{E}\ \left[\left\|g_h(\psi^{r,k-1,m}; \bar{z}_{h+1}^{r,k-1,m}) - \mathbb{E}_{\bar{Z}_{h+1}^{r,k-1,m}}\ g_h(\psi^{r,k-1,m}; \bar{z}_{h+1}^{r,k-1,m})\right\|_2^2\right] \\
\leq&\sum_h \mathbb{E}\left[\left\|\psi_h^{r,k-1,m} - \widetilde{\psi}_h^r + \alpha^r \mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}}\ g_h(\psi^{r,k-1,m}; \bar{z}_{h+1}^{r,k,m})\right\|_2^2\right] + (\alpha^r)^2 \cdot H\sigma_g^2
\end{aligned}$$

The third step is due to the cross term being zero if we condition on $\mathcal{F}^{r,k-1,m}$. We proceed by telescope again to replace the gradients at each parameter $\psi^{r,k,m}$ with that at the last synchronization $\widetilde{\psi}^r$.

$$\begin{aligned}
&\underbrace{\mathbb{E}\left[\left\|\psi^{r,k,m} - \widetilde{\psi}^r\right\|_2^2\right]}_{\text{Local drift}} \\
\leq&(\alpha^r)^2 \cdot H\sigma_g^2 + (\alpha^r)^2\left(1 + \frac{1}{\xi}\right)\left\|\mathbb{E}_{Z_{h+1}}\ g(\widetilde{\psi}^r; Z_{h+1})\right\|_2^2 \\
&+ (1+\xi)\sum_h \mathbb{E}\left[\|\|\underbrace{\psi_h^{r,k-1,m} - \widetilde{\psi}_h^r}_{\text{I}} + \underbrace{\alpha^r \mathbb{E}_{\bar{Z}_{h+1}^{r,k,m}}\ g_h(\psi^{r,k-1,m}; \bar{z}_{h+1}^{r,k,m}) - \alpha^r \mathbb{E}_{Z_{h+1}}\ g_h(\widetilde{\psi}^r; Z_{h+1})}_{\text{II}}\|\|_2^2\right]
\end{aligned}$$

This relation holds for any positive ξ due to Young's inequality (Fact 2) and later we will select an appropriate value of ξ that balances the terms when we unroll k to 0. Next, we associate term II with term I by the 4-Lipschitzness of the semi-gradient stated in Lemma 9:

$$\left\|\mathbb{E}_{\bar{Z}_{h+1}}\ g(\psi^1; \bar{Z}_{h+1}) - \mathbb{E}_{\bar{Z}_{h+1}}\ g(\psi^2; \bar{Z}_{h+1})\right\|_2 \leq 4\left\|\psi^1 - \psi^2\right\|_2$$

The cross term between I and II can be derived from Eq. (25) and Lemma 10.

$$\mathbb{E}\left\langle \psi^1 - \psi^2 \ , \ \mathbb{E}_{\bar{Z}}\left[g(\psi^1;\bar{Z})\right] - \mathbb{E}_{\bar{Z}}\left[g(\psi^2;\bar{Z})\right]\right\rangle \le -(1-\gamma)\mathbb{E}\ \mathcal{E}(\psi^1;\psi^2) \le -(1-\gamma)\mu_\phi \cdot \mathbb{E}\left\|\psi^1-\psi^2\right\|_2^2$$

provided that the feature covariate matrix is positive-definite. By rearranging terms and then taking the ensemble mean, we see that the drift term subjects to a linear dynamic equation:

$$\text{drift}(k;r) \le A\cdot \text{drift}(k;r) + B \tag{32}$$

where

$$\begin{aligned} \text{drift}(k;r) :=& \frac{\sum_m}{M}\mathbb{E}\left[\left\|\psi^{r,k,m} - \widetilde{\psi}^r\right\|_2^2\right] \\ A :=& (1+\xi)\left(1+16(\alpha^r)^2 - 2\alpha^r(1-\gamma)\mu_\phi\right) \\ B :=& (\alpha^r)^2\cdot\left(H\sigma_g^2 + \left(1+\frac{1}{\zeta}\right)G^2\right) \end{aligned} \tag{33}$$

The synchronization rule (line 16 in Algorithm 2) suggests that $\text{drift}(0;r)=0$, which then implies

$$\text{drift}(k;r) \le A^k\text{drift}(0;r) + B\cdot\sum_{t=0}^{k}A^t = A^k\cdot 0 + B\cdot\sum_{t=0}^{k}A^t \le kA^k\cdot B$$

It is also convenient to abbreviate the local drift as $\text{drift}(r)$. and the definitions of the drift terms naturally suggests that

$$\text{drift}(r) = \underbrace{\frac{\sum_{m,k}}{MK}\mathbb{E}\left[\left\|\psi^{r,k,m}-\widetilde{\psi}^r\right\|_2^2\right]}_{\text{Local drift}} = \frac{1}{K}\sum_{k=0}^{K-1}\underbrace{\text{drift}(k;r)}_{\text{Ensemble drift}} < B\cdot\frac{1}{K}\sum_{k=0}^{K-1} k\cdot A^k \tag{34}$$

We will compute the right-hand-side of Eq. (34) and then complete the proofs.

First we would like to simplify the scaling term A in Eq.(32) and sure that unrolling k will not cause any terms to diverge when K is large. Pick $\xi=\frac{1}{K}$ and $\alpha^r < \frac{\mu_\phi(1-\gamma)}{16}$, we can guarantee that $(1+\xi)^K < e$ and that $\left(1+16(\alpha^r)^2-2\alpha^r(1-\gamma)\mu_\phi\right) \le 1-(1-\gamma)\mu_\phi\alpha^r \le e^{-(1-\gamma)\mu_\phi\alpha^r}$ Next we connect the summation to a Riemannian integral and provide sharp bounds for all $K\in\mathbb{Z}_+$:

$$\begin{aligned} \text{drift}(r) <& eB\cdot\frac{1}{K}\int_0^K xe^{-ax}dx\Big|_{a=(1-\gamma)\mu_\phi} = eB\cdot\frac{1-(1+aK)\cdot e^{-aK}}{a^2K}\Big|_{a=(1-\gamma)\mu_\phi} \\ =& \Gamma(\gamma,\mu_\phi,K)\cdot\frac{B}{K} \le \begin{cases} \frac{eB}{(1-\gamma)^2{\mu_\phi}^2K} & a>0 \\ \frac{eBK}{2} & a\to 0^+ \end{cases} \end{aligned}$$

For reader's reference, we plot the summation with its upper bounds evaluated at $\mu_\phi(1-\gamma)=0.07$: which

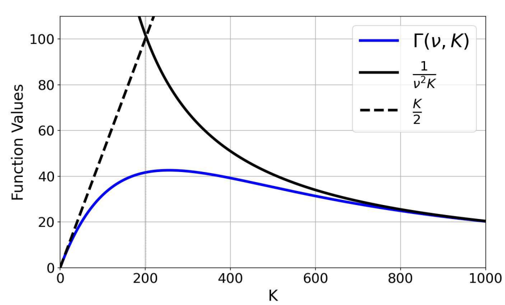

Figure 3: Upper bounds on drift(r) across different K. Curves evaluated at $B=1, \mu_\phi(1-\gamma)=0.007$

clearly shows that the bound $\frac{eB}{(1-\gamma)^2{\mu_\phi}^2K}$ is approximately effective when K is large, while $\frac{eBK}{2}$ dominates the summation when K or $\mu_\phi(1-\gamma)$ is small.

We complete the proof by substituting B with its definition in Eq. (33), where we select $\xi=\frac{1}{K}$. We will also need to use the simple fact that $\left(1+\frac{1}{K}\right)<2$. $\square$

B.1.4 Proof of Lemma 6

Proof. First we invoke a standard result in federated RL that links the gradient-dependent terms in Lemma 4 and Lemma 3. Readers may refer to Eq.(a) in the proofs of Proposition 4 and the second in equality for Lemma 10 in [53] for details. With Definition 1 and Lemma 9, we have

$$\mathbb{E}\left\langle \widetilde{\psi}^r - \psi^\star, \mathbb{E}_z g\left(\widetilde{\psi}^r; Z\right)\right\rangle + \eta_r \alpha_r K \cdot 2\left\|\mathbb{E}^{\pi}_{\bar{Z}}\ g\left(\widetilde{\psi}^r; \bar{Z}\right)\right\|_2^2 \leq -\mu_\phi(1-\gamma)U\left(\psi^r\right) + 16\mathbb{E}\ \mathcal{E}\left(\widetilde{\psi}^r\right)$$

Taking Lemma 3 and Lemma 4 to Eq. (19),

$$\begin{aligned} U(\widetilde{\psi}^{r+1}) =& U(\widetilde{\psi}^r) + \underbrace{2\mathbb{E}\left\langle \widetilde{\psi}^r - \psi^\star\ ,\ \widetilde{\psi}^{r+1} - \widetilde{\psi}^r\right\rangle}_{\text{Progression Direction}} + \underbrace{\mathbb{E}\left\|\widetilde{\psi}^{r+1} - \widetilde{\psi}^r\right\|_2^2}_{\text{Progression Distance}} \\ \leq& \left(1 - \frac{\eta_r\alpha_r K}{c}\right)U(\widetilde{\psi}^r) + \eta_r\alpha_r K \cdot \left[\left(-2(1-\gamma)\mu_\phi + \frac{2}{c}\right)\cdot U\left(\psi^r\right) + 64\eta_r\alpha_r K\mathbb{E}\mathcal{E}\left(\widetilde{\psi}^r\right)\right] \\ &+ \eta_r\alpha_r K\left(c + 4\eta_r\alpha_r K\right)\cdot \underbrace{16\frac{\sum_{m,k}}{MK}\mathbb{E}\left[\left\|\psi^{r,k,m} - \widetilde{\psi}^r\right\|_2^2\right]}_{\text{Local drift}} + 2(\eta_r\alpha_r K)^2\frac{H\sigma_g^2}{MK} \end{aligned} \tag{35}$$

Next, we further simplify the two terms in the middle, at the cost of several restrictions on the learning rates. First we would like to simplify the second term. With Eq. (26), we discover that as long as the composite learning rates satisfy $\eta_r\alpha_r K \leq \frac{\mu_\phi(1-\gamma)}{128}$, we are able to do suppress the Bellman error with the potential function, under the setting $c = \frac{2}{\mu_\phi(1-\gamma)}$:

$$-\mu_\phi(1-\gamma)\cdot U\left(\psi^r\right) + 64\cdot\eta_r\alpha_r K\mathbb{E}\mathcal{E}\left(\widetilde{\psi}^r\right) \leq \frac{-\mu_\phi(1-\gamma)}{2}\mathbb{E}\mathcal{E}\left(\widetilde{\psi}^r\right) \tag{36}$$

Under the requirement of $\alpha_r < \frac{\mu_\phi(1-\gamma)}{16}$, Lemma 5) suggest that the local drift term is dominated by

$$\underbrace{16\frac{\sum_{m,k}}{MK}\mathbb{E}\left[\left\|\psi^{r,k,m} - \widetilde{\psi}^r\right\|_2^2\right]}_{\text{Local drift}} \leq 16\Gamma(K;\nu_\phi)\left(\frac{H\sigma_g^2}{K} + 2G^2\right)\cdot\alpha_r^2$$

where $\Gamma(K;\nu_\phi)$ is defined in Eq. (31). We decrease the local learning rate with a factor of H to offset the non-stationarity of POMDPs.

With these two simplifications, we arrive at

$$\begin{aligned} U(\widetilde{\psi}^{r+1}) =& U(\widetilde{\psi}^r) + \underbrace{2\mathbb{E}\left\langle \widetilde{\psi}^r - \psi^\star\ ,\ \widetilde{\psi}^{r+1} - \widetilde{\psi}^r\right\rangle}_{\text{Progression Direction}} + \underbrace{\mathbb{E}\left\|\widetilde{\psi}^{r+1} - \widetilde{\psi}^r\right\|_2^2}_{\text{Progression Distance}} \\ \leq& \left(1 - \frac{\eta_r\alpha_r K}{2}\nu_\phi\right)U(\widetilde{\psi}^r) - \eta_r\alpha_r K\cdot\frac{\nu_\phi}{2H}\mathbb{E}\ \mathcal{E}\left(\widetilde{\psi}^r\right) \\ &+ (\eta_r\alpha_r K)^2\left(\frac{2}{\alpha_r K\nu_\phi} + 4\right)\cdot 16\Gamma(K;\nu_\phi)\left(\frac{H\sigma_g^2}{K} + 2G^2\right)\cdot\alpha_r^2 \\ &+ 2(\eta_r\alpha_r K)^2\frac{H\sigma_g^2}{MK} \end{aligned} \tag{37}$$

under the condition that $c = \frac{2}{\mu_\phi(1-\gamma)}$, $\nu_\phi = \mu_\phi(1-\gamma)$, $\alpha_r \leq \frac{\mu_\phi(1-\gamma)}{16}$ and $\eta_r\alpha_r K \leq \frac{\mu_\phi(1-\gamma)}{128}$. If further require that $\alpha_r \leq \frac{\mu_\phi(1-\gamma)}{16RK}$, then we could obtain $\left(\frac{2}{\alpha_r K\nu_\phi} + 4\right)\alpha_r^2 \leq \frac{9}{64RK^2}$ and arrive at

$$U\left(\widetilde{\psi}^{r+1}\right) \leq \left(1 - \frac{\iota_r}{2}\nu_\phi\right)U\left(\widetilde{\psi}^r\right) - \frac{\iota_r}{2}\nu_\phi\mathbb{E}\ \mathcal{E}\left(\widetilde{\psi}^r\right) + \frac{9}{4}\Gamma(K;\nu_\phi)\left(\frac{H\sigma_g^2}{RK^3} + \frac{2G^2}{RK^2}\right)\iota_r^2 + \frac{2H\sigma_g^2}{MK}\iota_r^2$$

□

B.2 Proof of Theorem 1

Proof. To further abbreviate expressions in Lemma 6, in what follows we write $\eta_r\alpha_r K$ as ι_r, $\frac{\nu_\phi}{2}\mathbb{E}\ \mathcal{E}(\widetilde{\psi}^r)$ as e_r, $U(\widetilde{\psi}^r)$ as u_r and A as $\frac{9}{4}\Gamma(K;\nu_\phi)\left(\frac{H\sigma_g^2}{RK^3}+\frac{2G^2}{RK^2}\right)+\frac{2H\sigma_g^2}{MK}$. Under the constraints that $\alpha_r \le \frac{\mu_\phi(1-\gamma)}{16RK}$, $\eta_r\alpha_r K = \iota_r \le \frac{\mu_\phi(1-\gamma)}{128}$, we obtain

$$e_r \le \left(\frac{1}{\iota_r}-\frac{\nu_\phi}{2}\right)u_r - \frac{1}{\iota_r}u_{r+1} + A\iota_r \tag{38}$$

Now we set the stochastic averaging weight as $w_r=(r+2)^2$, total weight as $W=\sum_{r=0}^R w_r \ge R^3/3$, composite learning rate as $\iota_r = \frac{a}{\mu_\phi(1-\gamma)(r+a)}$ where a is any positive constant larger than $\frac{9}{2}$. Then when the total rounds of communication is sufficiently large such $\forall R\ge R_0$, we ensure that $\iota_r \le \frac{\nu_\phi}{128}$, the weighted average of the last error terms is controlled by

$$\begin{aligned}
\frac{1}{W}\sum_{r=R_0}^{R} w_r e_r \le & \frac{3\nu_\phi}{R^3}\left(1-\frac{1}{a}\right)u_0 + \frac{3a}{\nu_\phi}\left(\frac{1}{2R}+(a-2)^2\frac{\ln(1+\frac{R+1}{a})}{R^3}\right)A \\
=&\widetilde{O}\left(\frac{\nu_\phi}{R^3}\left\|\bar{\psi}^0-\psi^\star\right\|_2^2+\frac{H\sigma_g^2}{\nu_\phi MKR}+\Gamma(K;\nu_\phi)\frac{H\sigma_g^2}{\nu_\phi R^2K^3}+\Gamma(K;\nu_\phi)\frac{G^2}{\nu_\phi R^2K^2}\right) \\
\le&\widetilde{O}\left(\frac{\nu_\phi Hd}{R^3}+\frac{H\sigma_g^2}{\nu_\phi MKR}+\Gamma(K;\nu_\phi)\frac{H\sigma_g^2}{\nu_\phi R^2K^3}+\Gamma(K;\nu_\phi)\frac{G^2}{\nu_\phi R^2K^2}\right)
\end{aligned}$$

The proofs of the first inequality is similar to Lemma 3.4 in [46] and we omit it for brevity. The last display is due to Eq. (24). By the convexity of the Bellman error, we conclude:

$$\mathbb{E}\ \mathcal{E}\left(\widehat{\psi}^R\right) \le \widetilde{O}\left(\frac{Hd}{R^3}+\frac{1}{{\nu_\phi}^2}\left[\left(\frac{1}{MT}+\frac{\Gamma(K;\nu_\phi)}{KT^2}\right)H\sigma_g^2+\frac{\Gamma(K;\nu_\phi)}{T^2}G^2\right]\right) \tag{39}$$

where we write the total number of gradient steps or sample trajectories taken by each parallel machine as $T=RK$. We remind the readers that the error bound holds true when $\alpha_r \le \frac{\mu_\phi(1-\gamma)}{16RK}$, $\eta_r = \frac{\iota_r}{\alpha_r K}$, $\iota_r = \frac{a}{\mu_\phi(1-\gamma)(r+a)}$, $w_r=(r+2)^2$ and we take the weighted average after $r \ge \frac{128a}{{\nu_\phi}^2}$. We complete the proof by the fact that $\Gamma(K;\nu_\phi) \le O\left(\min\left\{\frac{1}{{\nu_\phi}^2KT^2}, K^2\right\}\right)$ and $T=KR$.

B.3 Proof of Lemma 1

In what follows we design a parallel training setup to make the error bound in Theorem 1 be dominated by the varince reduction term, so that we will benefit from parallelism via a linear speedup in sample complexity. Simple calculation reveals that Eq. (39) reduces to $\mathbb{E}\ \mathcal{E}\left(\widehat{\psi}^R\right) \le \widetilde{O}\left(\frac{\frac{1}{{\nu_\phi}^2}H\sigma_g^2+Hd}{MT}\right)$ under the following conditions

Definition 14 (Linear Speedup Condition)**.**

$$K \le \frac{T^{2/3}}{M^{1/3}}, \quad \frac{\Gamma(K;\nu_\phi)}{K} < \frac{T}{M}, \quad \textit{and} \quad \Gamma(K;\nu_\phi) < \frac{H\sigma_g^2 T}{G^2 M} \tag{40}$$

The sample complexity to obtain an ϵ-accurate Q-function estimate is $N_{\text{eval}} \le \widetilde{O}\left(\frac{1}{M\epsilon}\left(\frac{H\sigma_g^2}{{\nu_\phi}^2}+Hd_Q\right)\right)$ trajectories, or $\widetilde{O}\left(\frac{H}{M\epsilon}\left(\frac{H\sigma_g^2}{{\nu_\phi}^2}+Hd_Q\right)\right)$ observation-action tuples.

To satisfy the constraints in Eq. (40), we can select the communication interval K according to the ill-condition number ν_ϕ, using the analysis results in Section B.1.3:

- When ν_ϕ is small, meaning that the feature matrix is ill-defined, we need to carefully limit the number of local steps to suppress local drift. We recommend choosing local steps

$$K \le \min\left\{\frac{T^{2/3}}{M^{1/3}}\ ,\ \frac{2}{e}\frac{T}{M}\ ,\ \sqrt{\frac{2}{e}\cdot\frac{H\sigma_g^2}{G}\cdot\frac{T}{M}},\right\} \tag{41}$$

so that the constraints in Eq. (40) is met using the fact that $\Gamma(K;\nu_\phi) \le \frac{eK^2}{2}$. Moreover, in this scenario the error bound in Eq. (39) is controlled via

$$\mathbb{E}\,\mathcal{E}\left(\widehat{\psi}^R\right) \le \widetilde{O}\left(\frac{Hd^2}{MT} + \frac{1}{{\nu_\phi}^2}\left[\left(\frac{1}{MT}+\frac{1}{RT}\right)H\sigma_g^2 + \frac{1}{R^2}G^2\right]\right) \le \widetilde{O}\left(\frac{Hd}{MT} + \frac{H\sigma_g^2}{{\nu_\phi}^2 MT}\right) \tag{42}$$

and the communication complexity is

$$R = \frac{T}{K} \ge \max\left\{(MT)^{1/3}\ ,\ \frac{e}{2}M\ ,\ \sqrt{\frac{e}{2}\frac{G}{H\sigma_g^2}MT}\right\} \tag{43}$$

- When ν_ϕ is large, it implies that the feature matrix is well-defined, we need to ensure that the total and local compuation budgets are large enough to keep the last two terms in Eq. (39) quickly converges. When that happends, the variance reduction term will stands out of our bound, which exhibits the benefits of parallelism by a linear speedup. To this end, we discover that a) $K \le \frac{T^{2/3}}{M^{1/3}}$ implies that $\frac{1}{R^3} \le \frac{1}{MT}$, b) $K \ge \frac{1}{{\nu_\phi}^2}\frac{M}{T}$ implies that $\frac{\Gamma(K;\nu_\phi)}{KT^2} \le \frac{e}{{\nu_\phi}^2KT^2} \lesssim \frac{1}{MT}$ and c) $\frac{T}{M} \ge \frac{1}{{\nu_\phi}^2}\frac{G^2}{H\sigma_g^2}$ impliese that $\frac{\Gamma(K;\nu_\phi)}{T^2}G^2 \le \frac{e}{{\nu_\phi}^2T^2}G^2 \lesssim \frac{H\sigma_g^2}{MT}$. When *a*) to *c*) are simultaneously satisfied, the sample complexity will scale with $\frac{1}{MT}$. One possible scenarios is to pick $K = \frac{T^{2/3}}{M^{1/3}}$ when $\frac{T}{M} \ge \frac{1}{{\nu_\phi}^2}\frac{G^2}{H\sigma_g^2}$. This choice directly meets the requirements of a) and c), while also ensures b) $\frac{T}{M} \ge \frac{1}{{\nu_\phi}^2K}$ since second-moment is larger than the variance. Under this setup,

$$\mathbb{E}\,\mathcal{E}\left(\widehat{\psi}^R\right) \le \widetilde{O}\left(\frac{Hd^2}{MT} + \frac{1}{{\nu_\phi}^2}\left[\left(\frac{1}{MT}+\frac{1}{{\nu_\phi}^2KT^2}\right)H\sigma_g^2 + \frac{1}{{\nu_\phi}^2T^2}G^2\right]\right) \le \widetilde{O}\left(\frac{H\sigma_g^2 + Hd}{{\nu_\phi}^2 MT}\right) \tag{44}$$

where the number of communications $R = \frac{T}{K} = (MT)^{1/3}$ improves Eq. (43), at the cost of a slightly increase Bellman error.

□

C Distributed NPG for POMDPs

C.1 Proof of Auxiliary Lemmas

Corollary 5 (β-smoothness)**.** *Assumption 5 implies*

$$(\theta_t - \theta_{t+1})^\top \nabla_\theta \ln \pi_h^{\theta_{t+1}}(a_h|z_h) - \frac{\beta}{2}\left\|\theta_t - \theta_{t+1}\right\|_2^2 \le \ln \pi_h^{\theta_t}(a_h|z_h) - \ln \pi_h^{\theta_{t+1}}(a_h|z_h) \tag{45}$$

The exact value of β is relevant to the policy class.

C.1.1 Proof of Lemma 1

We will show that

$$\mathsf{V}^{\pi^\star} - \mathbb{E}\frac{1}{R}\sum_{r=0}^{R-1}\mathsf{V}^{\pi^{\theta_r}} \le HW\sqrt{\frac{2\beta D}{R}} + \sqrt{\kappa H}\sqrt{\frac{1}{R}\sum_{r=0}^{R-1}\mathbb{E}\,\mathcal{L}_\mathsf{A}\left(\omega_r;\theta_r,\pi^{\theta_r}\right)}$$

Here, $\mathcal{L}_{\mathsf{A}}$ is the compatible function approximation error, defined in Eq. (10). κ is a regularity constant introduced in Eq. (9). The following proof is a smooth generalization of NPG regret bounds to POMDPs, which is inspired by [2, 9]. Our proof adopts a different definition for κ compared with [9], and our policy class is no longer limited to softmax parameterization like [9], or Markov policies like [2].

Proof. We would like to introduce some functions for the convenience of our analysis. First, let us define

$$\epsilon^\star(\omega_r;\theta_r) := \mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{A}_h^{\pi^{\theta_r}}\left(\bar{Z}_h\right) - \omega_r^\top\nabla_\theta \ln\pi_h^{\theta_r}(A_h|Z_h)\right) \tag{46}$$

We also define the following potential function under assumptions 5 and 9,

$$U(\pi^{\theta_r};\pi^\star) := \mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h D_{\mathrm{KL}}\left(\pi_h^\star\left(\cdot\mid Z_h\right)\|\pi_h^r\left(\cdot\mid Z_h\right)\right) < +\infty \tag{47}$$

which will be used as a tool to link the functional suboptimality $\mathsf{V}^{\pi^\star} - \mathsf{V}^{\pi^{\theta_r}}$ with the compatible function approximation error. We will also invoke Cauchy-Schwartz inequality for vectors $\sum_i a_i b_i \le \sqrt{\sum_i a_i^2}\cdot\sqrt{\sum_i b_i^2}$ and random variables $\mathbb{E}|XY| \le \sqrt{\mathbb{E}X^2}\sqrt{\mathbb{E}Y^2}$.

Conditioned on all the parameters θ_r,ω_r, we observe that

$$\begin{aligned}
&U\left(\pi^{\theta_r};\pi^\star\right) - U\left(\pi^{\theta_{r+1}};\pi^\star\right)\\
=&\mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\left[D_{\mathrm{KL}}\left(\pi_h^\star\left(\cdot\mid Z_h\right)\|\pi_h^{\theta_r}\left(\cdot\mid Z_h\right)\right) - D_{\mathrm{KL}}\left(\pi_h^\star\left(\cdot\mid Z_h\right)\|\pi_h^{\theta_{r+1}}\left(\cdot\mid Z_h\right)\right)\right]\\
=&\mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\int_{\mathsf{A}} dA_h\cdot\pi_h^\star(A_h|Z_h)\left(\ln\pi_h^{\theta_{r+1}}(A_h|Z_h) - \ln\pi_h^{\theta_r}(A_h|Z_h)\right)\\
\ge&\mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\left((\theta_{r+1}-\theta_r)\nabla_\theta\ln\pi_h^{\theta_r}(A_h|Z_h) - \frac{\beta}{2}\left\|\theta_r-\theta_{r+1}\right\|_2^2\right)\quad //\beta\text{-smoothness, Corollary 5}
\end{aligned} \tag{48}$$

$$\begin{aligned}
=&\eta_r\cdot\mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\left(\omega_r^\top\nabla_\theta\ln\pi_h^{\theta_r}(A_h|Z_h) - \mathsf{A}_h^{\pi^{\theta_r}}\left(\bar{Z}_h\right)\right) + \eta_r\cdot\mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\mathsf{A}_h^{\pi^{\theta_r}}\left(\bar{Z}_h\right)\\
&-\mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\frac{\beta}{2}\eta_r^2\left\|\omega_r\right\|_2^2\quad //\text{Telescope the advantage function and invoke } \theta_{r+1}=\theta_r+\eta_r\omega_r\\
=&-\eta_r\cdot\epsilon^\star(\omega_r;\theta_r) + \eta_r\left(\mathsf{V}^{\pi^\star} - \mathsf{V}^{\pi^{\theta_r}}\right) - \frac{\beta W^2 H}{2}\eta_r^2\quad //\text{Performance difference lemma 7}
\end{aligned} \tag{49}$$

Dividing both sides of Eq. (49) with η_r, we obtain the following fact when conditioned on θ_r and ω_r:

$$\mathsf{V}^{\pi^\star} - \mathsf{V}^{\pi^{\theta_r}} \le \frac{U(\pi^{\theta_r};\pi^\star) - U(\pi^{\theta_{r+1}};\pi^\star)}{\eta_r} + \epsilon^\star(\omega_r;\theta_r) + \frac{\beta W^2 H}{2}\eta_r \tag{50}$$

Fix $\eta_r \equiv \eta$ for all r. Conditioned on all the parameters θ_r,ω_r returned by SGD, we take the average over all

r and obtain

$$
\begin{aligned}
\frac{1}{R}\sum_{r=0}^{R-1}\mathsf{V}^{\pi^\star}-\mathsf{V}^{\pi^{\theta_r}} \leq & \frac{1}{\eta R}\sum_{r=0}^{R-1}U(\pi^{\theta_r};\pi^\star)-U(\pi^{\theta_{r+1}};\pi^\star)+\frac{1}{R}\sum_{r=0}^{R-1}\epsilon^\star(\omega;\theta_r)+\frac{\beta W^2 H}{2}\eta \\
= & \frac{\mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\left[D_{\mathrm{KL}}\left(\pi_h^\star(\cdot\mid Z_h)\,\|\pi_h^0(\cdot\mid Z_h)\right)-D_{\mathrm{KL}}\left(\pi_h^\star(\cdot\mid Z_h)\,\|\pi_h^R(\cdot\mid Z_h)\right)\right]}{\eta R} \\
& +\frac{1}{R}\sum_{r=1}^{R}\epsilon^\star(\omega;\theta_r)+\frac{\beta W^2 H}{2}\eta \quad //U \text{ is non-negtive} \\
\leq & \frac{HD}{\eta R}+\frac{\beta W^2 H}{2}\eta+\frac{1}{R}\sum_{r=0}^{R-1}\epsilon^\star(\omega_r;\theta_r) \quad // \text{ Definition 4.} \\
\leq & HW\sqrt{\frac{2\beta D}{R}}+\frac{1}{R}\sum_{r=0}^{R-1}\epsilon^\star(\omega_r;\theta_r) \quad //\text{Optimize } \eta_r \text{ over } \mathbb{R} \text{ to get } \eta=\frac{\sqrt{2D/\beta}}{W}\frac{1}{\sqrt{R}}
\end{aligned}
\tag{51}
$$

Next, we connect the term $\epsilon^\star(\omega;\theta_r)$ with the compatible function approximation error introduced in Eq. (10):

$$
\begin{aligned}
& \epsilon^\star(\omega_r;\theta_r) \\
:= & \mathbb{E}^{\pi^\star}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{A}_h^{\pi^{\theta_r}}\left(\bar{Z}_h\right)-\omega_r{}^\top\nabla_\theta\ln\pi_h^{\theta_r}\left(A_h\mid Z_h\right)\right) \quad //\text{Definition in Eq. (46)} \\
= & \mathbb{E}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^{\frac{h}{2}}\frac{d\mathbb{P}^{\pi^\star}}{d\mathbb{P}^{\pi^{\theta_r}}}\gamma^{\frac{h}{2}}\left(\mathsf{A}_h^{\pi^{\theta_r}}\left(\bar{Z}_h\right)-\omega_r{}^\top\nabla_\theta\ln\pi_h^{\theta_r}\left(A_h\mid Z_h\right)\right) \\
\leq & \mathbb{E}^{\pi^{\theta_r}}\sqrt{\sum_{h=0}^{H-1}\gamma^h\left(\frac{d\mathbb{P}^{\pi^\star}}{d\mathbb{P}^{\pi^{\theta_r}}}\right)^2}\sqrt{\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{A}_h^{\pi^{\theta_r}}\left(\bar{Z}_h\right)-\omega_r{}^\top\nabla_\theta\ln\pi_h^{\theta_r}\left(A_h\mid Z_h\right)\right)^2} \\
& //\text{Cauchy-Schwartz for vectors} \\
\leq & \sqrt{\mathbb{E}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\frac{d\mathbb{P}^{\pi^\star}}{d\mathbb{P}^{\pi^{\theta_r}}}\right)^2}\sqrt{\mathbb{E}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{A}_h^{\pi^{\theta_r}}\left(\bar{Z}_h\right)-\omega_r{}^\top\nabla_\theta\ln\pi_h^{\theta_r}\left(A_h\mid Z_h\right)\right)^2} \\
& //\text{Cauchy-Schwartz for random variables} \\
= & \sqrt{\mathbb{E}^{\pi^\star}\frac{d\mathbb{P}^{\pi^\star}}{d\mathbb{P}^{\pi^{\theta_r}}}\sum_{h=0}^{H-1}\gamma^h}\sqrt{\mathbb{E}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{A}_h^{\pi^{\theta_r}}\left(\bar{Z}_h\right)-\omega_r{}^\top\nabla_\theta\ln\pi_h^{\theta_r}\left(A_h\mid Z_h\right)\right)^2} \\
\leq & \sqrt{\kappa_r H}\cdot\sqrt{\mathcal{L}_{\mathsf{A}}\left(\omega_r;\theta_r,\pi^{\theta_r}\right)} \quad //\text{Definition of } \kappa_r \text{ in Assumption (9) and } \mathcal{L}_{\mathsf{A}} \text{ in Eq. (10).}
\end{aligned}
\tag{52}
$$

Taking the expectation w.r.t. $\theta_{0:R-1}$ and $\omega_{0:R-1}$, we obtain

$$
\begin{aligned}
& \frac{1}{R}\sum_{r=0}^{R-1}\ \mathbb{E}\ \epsilon^\star(\omega_r;\theta_r) \\
\leq & \frac{\sum_{r=0}^{R-1}}{R}\sqrt{H\mathbb{E}\ \kappa_r}\sqrt{\mathbb{E}_{\omega_r,\theta_r}\mathcal{L}_{\mathsf{A}}(\omega_r;\theta_r,\pi^{\theta_r})} \quad //\text{Cauchy-Schwartz for random variables} \\
\leq & \sqrt{H}\sqrt{\frac{\sum_{r=0}^{R-1}}{R}\mathbb{E}\ \kappa_r}\cdot\sqrt{\frac{\sum_{r=0}^{R-1}}{R}\mathbb{E}\ \mathcal{L}_{\mathsf{A}}\left(\omega_r;\theta_r,\pi^{\theta_r}\right)} \quad //\text{Cauchy-Schwartz for vectors} \\
= & \sqrt{\kappa H}\cdot\sqrt{\frac{\sum_{r=0}^{R-1}}{R}\mathbb{E}\ \mathcal{L}_{\mathsf{A}}\left(\omega_r;\theta_r,\pi^{\theta_r}\right)} \quad //\text{Definition of } \kappa \text{ in Eq. (9)}
\end{aligned}
\tag{53}
$$

Bring Eq. (53) to Eq. (51), we conclude the proof. □

C.1.2 Discussion on κ

The authors of [9] introduced a similar concept of κ. In this work, we replace their definition with a finer characterization by investigating how the regularity constant evolves during policy improvement. Recall that we defined κ_r by $\kappa_r = \mathbb{E}^{\pi^\star} \frac{d\mathbb{P}^{\pi^\star}}{d\mathbb{P}^{\pi^{\theta_r}}}$ and referred to κ as $\kappa := \sqrt{\frac{1}{R}\sum_{r=1}^{R} \mathbb{E}\ \kappa_r}$. By expanding the trajectory probability along the horizon,

$$\begin{aligned}\mathbb{P}^{\pi^\theta}(\bar{Z}_H) =& \rho(S_0)\mathbb{O}_0(O_0|S_0)\pi_0^\theta(A_0|O_0)\mathbb{T}_1(S_1|S_0, A_0)\mathbb{O}_1(O_1|S_1)\pi_1^\theta(A_1|Z_1)\dots\\ &\mathbb{T}_{H-1}(S_{H-1}|S_{H-2}, A_{H-2})\mathbb{O}_{H-1}(O_{H-1}|S_{H-1})\pi_{H-1}^\theta(A_{H-1}|Z_{H-1})\end{aligned}$$

we relate the sample-path distribution $\mathbb{P}^{\pi^\theta}$ with the policy distribution: $\kappa_r = \mathbb{E}^{\pi^\star}\prod_{t=0}^{H-1}\frac{\pi_t^\star(A_t|Z_t)}{\pi_t^{\theta_r}(A_t|Z_t)} = \mathbb{E}^{\pi^\star}\exp\sum_{t=0}^{H-1}\ln\pi_t^\star(A_t|Z_t) - \ln\pi_t^{\theta_r}(A_t|Z_t)$. Using the β-smoothness of the policy, we can derive an upper bound on κ_r, even in continuous action spaces.

$$\kappa_r \leq \mathbb{E}^{\pi^\star}\exp\sum_{t=0}^{H-1}\beta\left\|\theta^\star - \theta_r\right\|_2 = \exp H\beta\left\|\theta^\star - \theta_r\right\|_2 \qquad \kappa \leq \mathbb{E}\sqrt{\frac{1}{R}\sum_{r=1}^{R}\exp H\beta\left\|\theta^\star - \theta_r\right\|_2} \tag{54}$$

Despite not considering the policy's structural properties, this bound still provides certain intuition. Notably, while this upper bound depends on H, it is unaffected by the sizes of the state and observation spaces, highlighting the algorithm's strong scalability for large-scale POMDP problems. We also notice that, as Natural Policy Gradient updates approach the optimal solution, κ_r converges to 1. However, suppressing κ_r below a small constant becomes more challenging as temporal information increases (H is large) or the policy class loses smoothness (β large).

Moreover, κ_r represents the exponent of the second-order Rényi divergence between optimal and learned sample path probabilities. By linking κ_r with f-divergences [3], we can establish upper and lower bounds on κ related to the geometry of the probability simplex. The following result is even independent of policy parameterization and applicable to potentially continuous action spaces.

$$\begin{aligned}\exp\sum_{t=0}^{H-1}\mathbb{E}^{\pi^\star}D_{KL}\left(\pi_t^\star(\cdot|Z_t)||\pi_t^{\theta_r}(\cdot|Z_t)\right) \leq \kappa_r \leq& 1 + \chi^2\left(\mathbb{P}^{\pi^\star}||\mathbb{P}^{\pi^{\theta_r}}\right)\\ \leq& 1 + \left\|\prod_{t=0}^{H-1}\frac{\pi^{\theta^\star}(A_t|Z_t)}{\pi^{\theta_r}(A_t|Z_t)}\right\|_\infty\sqrt{2\sum_{t=0}^{H-1}\mathbb{E}^{\pi^\star}D_{KL}\left(\pi_t^\star(\cdot|Z_t)||\pi_t^{\theta_r}(\cdot|Z_t)\right)}\end{aligned} \tag{55}$$

Here, $\chi^2(\cdot||\cdot)$ is the Chi-square divergence between two probability measures. The proof is a simple application of basic information-theoretic inequalities detailed in [15, 16], and we omit it for brevity. We observe that as the KL-divergence between the policies approaches zero, κ_r converges to 1. It is promising that applying regularization for POMDPs based on information theory can help reduce the upper bound on κ_r, which we leave for future work.

C.1.3 Proof of Lemma 2

We will show a stronger version of the Lemma: $\mathcal{L}_{\mathsf{A}}(\omega;\theta,\pi^\theta) \leq \mathcal{L}_{\mathsf{Q}}(\omega;\theta,\pi^\theta) \leq 2\cdot\epsilon_{\text{stat}}(\omega;\theta) + 2\cdot\epsilon_{\text{approx}}(\theta) + 4\cdot\mathcal{E}^{\pi^\theta}\left(\widehat{\psi}\right)$.

Proof. Under softmax parameterization, the policy takes the form of $\pi_h^\theta(a_h|z_h) := \frac{\exp f_\theta(z_h,a_h;h)}{\sum_{a_h\in\mathcal{A}}\exp f_\theta(z_h,a_h;h)}$ Consequently, the log-likelihood of the policy distribution is

$$\log\pi_h^\theta(a_h|z_h) = f_\theta(z_h, a_h; h) - \log\sum_{a_h\in\mathcal{A}}\exp f_\theta(z_h, a_h; h)$$

Taking the derivative to both sides, the score function of softmax policies is

$$\nabla_\theta\log\pi_h^\theta(a_h|z_h) = \nabla_\theta f_\theta(z_h, a_h; h) - \mathbb{E}_{a_h\sim\pi_h^\theta(\cdot|z_h)}\nabla_\theta f_\theta(z_h, a_h; h) \tag{56}$$

which can be viewed as the centered value of the un-normalized output of the neural network f_θ. Interestingly, this structure mirrors the relationship between the advantage function and the action-value function

$$\mathsf{A}_h^{\pi^{\theta_r}}(\bar{Z}_h) = \mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \mathbb{E}_{A_h\sim\pi^\theta(\cdot|Z_h)}\mathsf{Q}_h^{\pi^{\theta_r}}(Z_h, A_h) \quad a.s. \tag{57}$$

With these facts, we immediately conclude that the compatible function approximation error $\mathcal{L}_{\mathsf{A}}(\omega_r;\theta_r;\pi^{\theta_r})$ defined in Eq. (10) is dominated by the Q-compatible function approximation error $\mathcal{L}_{\mathsf{Q}}(\omega_r;\theta_r;\pi^{\theta_r})$ introduced in Eq. (11), under neural softmax parameterization.

$$\begin{aligned}
&\mathcal{L}_{\mathsf{A}}(\omega_r;\theta_r;\pi^{\theta_r})\\
=&\mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{A}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r^\top\nabla_\theta \ln \pi_h^{\theta_r}(A_h \mid Z_h)\right)^2 \quad //\text{Definition in Eq. (10)}\\
=&\mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\Big(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \mathbb{E}_{A_h\sim\pi^\theta(\cdot|Z_h)}\mathsf{Q}_h^{\pi^{\theta_r}}(Z_h, A_h)\\
&-\omega_r^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h) + \omega_r^\top\mathbb{E}_{A_h\sim\pi_h^\theta(\cdot|Z_h)}\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\Big)^2 \quad //\text{Eq. (56) and Eq. (57)}\\
=&\mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h) - \mathbb{E}_{A_h\sim\pi^\theta(\cdot|Z_h)}\left(\mathsf{Q}_h^{\pi^{\theta_r}}(Z_h,A_h) - \omega_r^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\right)\right)^2\\
=&\mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\,\mathrm{Var}_{A_h\sim\pi_h^{\theta_r}(\cdot|Z_h)}\left[\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\right]\\
\le&\mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\mathbb{E}_{A_h\sim\pi_h^{\theta_r}(\cdot|Z_h)}\left(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\right)^2 \quad //\text{Variance is less than second moment.}\\
=&\mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\right)^2 = \mathcal{L}_{\mathsf{Q}}(\omega_r;\theta_r;\pi^{\theta_r}) \quad //\text{Definition in Eq. (11)}
\end{aligned}$$

Next, we use several telescoping technique to associate the last display with the evaluation error. In the forthcoming derivations, we will write $\widehat{Q}_h^{\pi^{\theta_r}}$ as the approximated Q-function, and refer to the corresponding Bellman error

$$\mathcal{E}^{\pi^{\theta_r}}(\psi) := \frac{1}{2}\mathbb{E}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \widehat{\mathcal{Q}}_h^{\pi^{\theta_r}}(\bar{Z}_h)\right)^2 \tag{58}$$

as the evaluation error $\epsilon_{\text{eval}}(\theta_r)$. We will also invoke the definitions of the statistical and approximation error for Q-NPG algorithm, which we would like to repeat it here for the reader's convenience.

$$\epsilon_{\text{approx}}(\theta_r) := \inf_{\omega\in\mathbb{R}^{d_\theta}}\ \mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\widehat{\mathcal{Q}}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\right)^2 \tag{59}$$

$$\epsilon_{\text{stat}}(\omega_r;\theta_r) := \mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\widehat{\mathcal{Q}}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r{}^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\right)^2 - \epsilon_{\text{approx}}(\theta_r) \tag{60}$$

Now we are ready to state that,

$$\begin{aligned}
&\mathcal{L}_{\mathsf{Q}}(\omega_r;\theta_r;\pi^{\theta_r}) = \mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\right)^2\\
=&\mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}}\sum_{h=0}^{H-1}\gamma^h\left(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \widehat{\mathcal{Q}}_h^{\pi^{\theta_r}}(\bar{Z}_h) + \widehat{\mathcal{Q}}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r^\top\nabla_\theta f_{\theta_r}(Z_h,A_h;h)\right)^2 \quad //\text{Telescope a term of } \widehat{\mathcal{Q}}_h^{\pi^{\theta_r}}
\end{aligned}$$

$$\leq 2 \cdot \mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}} \sum_{h=0}^{H-1} \gamma^h \left(\mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \widehat{\mathcal{Q}}_h^{\pi^{\theta_r}}(\bar{Z}_h) \right)^2 + 2 \cdot \mathbb{E}_{\mathcal{P}}^{\pi^{\theta_r}} \sum_{h=0}^{H-1} \gamma^h \left(\widehat{\mathcal{Q}}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \omega_r^\top \nabla_\theta f_{\theta_r}(Z_h, A_h; h) \right)^2$$

$$// (a+b)^2 \leq 2a^2 + 2b^2$$

$$= 4 \cdot \epsilon_{\text{eval}}(\theta_r) + 2 \cdot (\epsilon_{\text{stat}}(\omega_r; \theta_r) + \epsilon_{\text{approx}}(\theta_r)) \quad // \text{Eq. (58), Eq. (59), and Eq. (60).}$$

Putting things together, we conclude

$$\mathcal{L}_{\mathsf{A}}(\omega_r; \theta_r; \pi^{\theta_r}) \leq \mathcal{L}_{\mathsf{Q}}(\omega_r; \theta_r; \pi^{\theta_r}) \leq 4\epsilon_{\text{eval}}(\theta_r) + 2\epsilon_{\text{stat}}(\omega_r; \theta_r) + 2\epsilon_{\text{approx}}(\theta_r) \tag{61}$$

□

Remark 10. Even though the compatible function approximation error relates to the advantage function, we don't need to calculate the advantage function directly from the Q-functions using $\mathsf{A}_h^{\pi^{\theta_r}}(\bar{Z}_h) = \mathsf{Q}_h^{\pi^{\theta_r}}(\bar{Z}_h) - \mathbb{E}_{A_h \sim \pi^\theta(\cdot|Z_h)} \mathsf{Q}_h^{\pi^{\theta_r}}(Z_h, A_h)$, as done in [9]. In fact, this conversion is possibly intractable, since it requires evaluating all possible actions for every historical input, which results in a high computational cost of $O\left(\mathcal{O}^H \mathcal{A}^H\right)$. To address this challenge, we can directly adopt a Q-function estimate and run Q-NPG instead, which still allows us to control the error of $\mathcal{L}_{\mathsf{A}}(\omega_r; \theta_r; \pi^{\pi^{\theta_r}})$, as shown in Eq. (61).

C.1.4 Lemma 11

We present a linear speedup guarantee for running a minibatch parallel SGD algorithm on smooth functions under the Poliak-Łojasiewicz condition.

Lemma 11 (Adapted from Theorem 1 of [20])**.** *Suppose the following assumptions hold: the stochastic gradients are unbiased and their variance evaluated on a minibatch of size B is bounded as $\mathbb{E}\left[\|\widetilde{\mathbf{g}} - \mathbf{g}\|^2\right] \leq C_1\|\mathbf{g}\|^2 + \frac{\sigma^2}{B}$, where C_1 and σ are non-negative constants, and the objective function $F(\mathbf{x})$ is differentiable and L-smooth: $\|\nabla F(\mathbf{x}) - \nabla F(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, and it satisfies the Polyak-Łojasiewicz (PL) condition with constant μ_F : $\frac{1}{2}\|\nabla F(\mathbf{x})\|_2^2 \geq \mu_F\left(F(\mathbf{x}) - F(\mathbf{x}^*)\right), \forall \mathbf{x} \in \mathbb{R}^d$ with $\mathbf{x}^*$ is an optimal solution, that is, $F(\mathbf{x}) \geq F(\mathbf{x}^*), \forall \mathbf{x}$. For LUPA-SGD [20] with τ local updates, if we choose the learning rate as $\eta_t = \frac{4}{\mu_F(t+a)}$ where $a = \alpha\tau + 4$ with α being a constant satisfying $\alpha \exp\left(-\frac{2}{\alpha}\right) < \kappa\sqrt{192\left(\frac{p+1}{p}\right)}$, choose mini-batch size as b, local step number as $K = O\left(\frac{T^{\frac{2}{3}}}{M^{\frac{1}{3}} b^{\frac{1}{3}}}\right)$, the functional suboptimality is controlled by*

$$\mathbb{E}\left[F\left(\bar{\mathbf{x}}_T\right)\right] - F^* \leq \frac{a^3\left(F\left(x_0\right) - F^*\right)}{(T+a)^3} + \frac{4L\sigma^2}{\mu_F^2 bM} \frac{1}{T+a} + \frac{256L^2\sigma^2}{\mu_F^3 bM} \frac{T(K-1)}{(T+a)^3} \lesssim \frac{L}{\mu_F^2} \frac{\sigma^2}{bMT} \tag{62}$$

Remark 11. For linear regression problems where $F(x) = \frac{\|Ax-b\|_2^2}{2}$, it is straightforward to evaluate that it satisfies the PL condition with $\mu_F = \lambda_{\min}\left(A^\top A\right)$. It is also L-smooth, with $L = \lambda_{\max}\left(A^\top A\right)$. [1] Notably, optimizing the compatible function approximation problem is also a least square regression. Specifically, $\underset{\omega}{\text{minimize}}\ \mathcal{L}_{\mathsf{A}}(\omega; \theta; \pi^\theta) = \underset{\omega}{\text{minimize}}\ \omega^\top \mathcal{F}_\theta\ \omega - 2\langle c, \omega \rangle$ where $c = \mathbb{E}^{\pi^\theta}\left[\sum_{h=0}^{H-1} \gamma^h \mathsf{A}^{\pi^\theta}(\bar{Z}_h) \nabla_\theta \ln \pi_h^{\pi^\theta}(A_h|Z_h)\right]$. According to Eq. (22), $c = \nabla_\theta \mathsf{V}^{\pi^\theta}$, which implies that

$$\underset{\omega}{\text{minimize}}\ \mathcal{L}_{\mathsf{A}}(\omega; \theta; \pi^\theta) = \underset{\omega}{\text{minimize}}\ \omega^\top \mathcal{F}_\theta\ \omega - 2\langle \nabla_\theta \mathsf{V}^{\pi^\theta}, \omega \rangle = \underset{\omega}{\text{minimize}}\ \frac{1}{2}\|A\omega - b\|_2^2$$

where A, b is any solution to $A^\top A = 2\mathcal{F}_\theta$ and $b^\top A = \nabla_\theta \mathsf{V}^{\pi^\theta}$. Consequently, the regularity constants for the compatible function approximation problem is $L = 2\lambda_{\max}\left(\mathcal{F}^{\pi^\theta}\right)$ and $\mu_F = 2\lambda_{\min}\left(\mathcal{F}^{\pi^\theta}\right)$.

[1] Here, we use $\lambda_{\min}(A)$ and $\lambda_{\max}(B)$ to represent the minumum and maximum eigenvalues of matrix A.

***Remark* 12.** When the Fisher information matrix is positive definite, direct computation reveals that the optimal solution $\omega_r^\star := \mathcal{F}_{\theta_r}^\top \nabla_\theta \mathsf{V}^{\pi^{\theta_r}}$ ensures

$$\begin{aligned}
\nabla_\theta \mathcal{L}_{\mathsf{A}}(\omega_r^\star; \theta_r, \pi^{\theta_r}) &= -2 \cdot \mathbb{E}^{\pi^{\theta_r}} \left[\sum_{h=0}^{H-1} \gamma^h \left(\mathsf{A}_h^{\pi^{\theta_r}} - \nabla_\theta^\top \ln \pi_h^{\theta_r}(A_h|Z_h)\omega_r^\star \right) \ln \pi_h^{\pi^{\theta_r}}(A_h|Z_h) \right] \\
&= -2 \cdot \mathbb{E}^{\pi^{\theta_r}} \left[\sum_{h=0}^{H-1} \gamma^h \left(\mathsf{Q}_h^{\pi^{\theta_r}} - \nabla_\theta^\top \ln \pi_h^{\theta_r}(A_h|Z_h)\omega_r^\star \right) \ln \pi_h^{\pi^{\theta_r}}(A_h|Z_h) \right] \\
&= -2 \cdot \left(\nabla_\theta \mathsf{V}^{\theta_r} - \mathcal{F}_{\theta_r}^\top \omega_r^\star \right) = 0
\end{aligned} \tag{63}$$

where the second step is due to Theorem 4.

C.2 Proof of Lemma 2

Proof. Let us first show the result concerning sample complexity. Under the convention that $T = RK$, if we pick $K = \frac{T^{1/2}}{M^{1/2}}$ (which implies $R = (MT)^{1/2}$), Theorem 2 implies that

$$\mathsf{V}^{\pi^\star} - \mathbb{E}_{\theta_r}\left[\frac{1}{R}\sum_{r=1}^{R} \mathsf{V}^{\pi^{\theta_r}}\right] \le \frac{HW\sqrt{2\beta D} + \frac{1}{\mu_F}\sqrt{\kappa H L \sigma_w^2}}{\sqrt{MK}} + \epsilon_{\text{app}} = O\left(\frac{1}{\sqrt{MK}}\right) + \epsilon_{\text{app}}$$

where we abbreviated the function approximation error $\frac{\sqrt{\kappa H}}{R}\sum_{r=1}^{R}\sqrt{\mathbb{E}\ \epsilon_{approx}(\theta_r)}$ as ϵ_{app}. Consequently, to ensure ϵ-suboptimality, the number of sample trajectories used in each round r is

$$N_{\text{per\_round}} = K \ge \frac{H^2W^2\beta D + \frac{1}{\mu_F}\kappa H L \sigma_w^2}{M(\epsilon - \epsilon_{\text{app}})^2} \approx \frac{H^2W^2\beta D + \frac{1}{\mu_F}\kappa H L \sigma_w^2}{M\epsilon^2}$$

provided that the function approximation error is small. Consequently, the total number of samples used in all the policy updates combined is

$$N_{\text{total}} = T = RK = MK^2 \gtrsim \frac{H^4W^4\beta^2D^2 + \frac{\kappa^2}{\mu_F^2}H^2L^2\sigma_w^2}{M\epsilon^4}$$

, exhibits a linear speedup w.r.t. M. For communication complexity, the number of times of gradient communication equals $R \cdot I = R \cdot O\left(\frac{T^{1/3}}{M^{2/3}}\right) = O\left(\frac{T^{5/6}}{M^{1/6}}\right)$ in each round, which is accompanied with $R = \sqrt{MT}$ times of synchronizations of the policy parameter. □

C.3 Proof of Theorem 3

Proof. According to Eq. (42), the Bellman error of policy evaluation is $\mathbb{E}\ \mathcal{E}\left(\widehat{\psi}^r; \theta_r\right)$ is controlled by $\widetilde{O}\left(\frac{H}{MT_e}\left(\frac{d_Q+\sigma_g^2}{{\nu_\phi}^2}\right)\right)$. We set the number of local steps for policy improvement $K = \sqrt{\frac{T}{M}}$, number of rounds of policy update $R = \sqrt{MT}$ and the number of steps for policy evaluation as $T_e = K$. This allows the functional suboptimality to be

$$\mathsf{V}^{\pi^\star} - \mathbb{E}_{\theta_{1:R}}\left[\frac{1}{R}\sum_{r=1}^{R} \mathsf{V}^{\pi^{\theta_r}}\right] \lesssim \frac{HW\sqrt{\beta D} + \frac{1}{\mu_F}\sqrt{\kappa H L \sigma_g^2} + \sqrt{\kappa H^2\left(\frac{d_Q+\sigma_g^2}{{\nu_\phi}^2}\right)}}{\sqrt{MK}} + \epsilon_{\text{app}}$$

To ensure ϵ-suboptimality, we need to ensure

$$K \gtrsim \frac{H^2W^2\beta D + \frac{1}{\mu_F^2}\kappa H L \sigma_g^2 + \kappa H^2\left(\frac{d_Q+\sigma_g^2}{{\nu_\phi}^2}\right)}{M(\epsilon - \epsilon_{\text{app}})^2} \approx \frac{H^2W^2\beta D + \frac{1}{\mu_F^2}\kappa H L \sigma_g^2 + \kappa H^2\left(\frac{d_Q+\sigma_g^2}{{\nu_\phi}^2}\right)}{M\epsilon^2}$$

provided that the function approximation error is small. These facts imply that the total sample complexity of Algorithm 1 is

$$N_{\text{total}} = N_{\text{eval}} + N_{\text{improve}} = 2T = 2MK^2 = O\left(\frac{C^2}{M\epsilon^4}\right)$$

where $C = H^2W^2\beta D + \frac{\kappa}{\mu_F^2} HL\sigma_g^2 + \kappa H^2 \left(\frac{d_Q + \sigma_g^2}{\nu_\phi{}^2}\right)$.

As for communication complexity, Algorithm 1 requires sending parameter $\theta \in \mathbb{R}^{d_\theta}$ and $\omega \in \mathbb{R}^{d_\theta}$ for $R + C = O\left(\sqrt{MT} + T^{5/6}M^{-1/6}\right)$ times, while communicating Q-network parameter $\psi \in \mathbb{R}^{d_Q}$ for $R \cdot (MT_e)^{1/3} = (MT)^{1/3}$. Putting things together, we need to communicate

$$P_{\text{total}} = O\left((MT)^{1/2} + T^{5/6}M^{-1/6}\right) d_\theta + (MT)^{1/3} \cdot d_Q$$

floating-point parameters. □